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Matrix factorisations and permutation branes

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Abstract: The description of B-type D-branes on a tensor product of two $N = 2$ minimal models in terms of matrix factorisations is related to the boundary state description in conformal field theory. As an application we show that the D0- and D2-brane for a number of Gepner models are described by permutation boundary states. In some cases (including the quintic) the images of the D2-brane under the Gepner monodromy generate the full charge lattice.

Keywords: Topological Strings, Conformal Field Models in String Theory, D-branes.

Contents

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1. Introduction

Recently, Maxim Kontsevich has suggested that supersymmetric B-type D-branes in Landau-Ginzburg models can be characterised in terms of matrix factorisations

$$Q^2 = W \cdot \mathbb{1}\,, \tag{1.1}$$

where $W(\Phi)$ is the Landau-Ginzburg superpotential of the superfield Φ. Here Q is an off-diagonal matrix

$$Q = \begin{pmatrix} 0 & J \\ E & 0 \end{pmatrix}, \tag{1.2}$$

and E and J are polynomial matrices in the superfield Φ. The matrices E and J appear in the action in the boundary F-terms, that also involve fermionic fields living at the boundary. Their presence is required in order to cancel a boundary term in the supersymmetry variation of the bulk F-term

$$\int_\Sigma d^2x\, d\theta^+ d\theta^-\, W + \text{c.c.}\,. \tag{1.3}$$

This approach was proposed in unpublished form by Kontsevich, and the physical interpretation of it was given in [1−6]; for a good review of this material see for example [7].

From a space-time point of view, these world-sheet fermions describe open string tachyons that appear in brane anti-brane configurations. (Recall that the open string GSO-projection for an open string between a brane and an anti-brane is opposite to that for the open string between two branes or two anti-branes; the above world-sheet fermions are projected out between branes and branes (or anti-branes and anti-branes), but survive the GSO-projection for open strings between branes and anti-branes.) Thus the matrix Q can also be thought of as describing the tachyonic configuration on (spacetime filling) brane anti-brane pairs that leads to the D-brane in question.

On the other hand, at least certain Landau-Ginzburg models have a microscopic description in terms of $N = 2$ minimal models [8−12]. These D-branes must therefore also have a conformal field theoretic description. It is then interesting to understand the relation between these two points of view in detail.

For the case of a single minimal model, this correspondence has been understood [2−4], but in general little is known. In this paper we want to study the next simple case,

$$W = x_1^d + x_2^d\,, \tag{1.4}$$

which will turn out to exhibit interesting and novel phenomena. One of the reasons why this case is likely to be significantly different comes from the fact that it has $c^{\text{total}} > 3$ (for $d \geq 5$). Regarded as a theory with respect to the diagonal $N = 2$ algebra (this is the symmetry seen by the matrix factorisation point of view), the theory is therefore not rational any more. Its brane spectrum will thus be much richer than for the case of a single minimal model.

Another motivation for looking at this example comes from the recent work of [13, 14]. They constructed a matrix factorisation both for a single D0-brane and a single D2-brane on the quintic. Geometrically, the D0 brane is described by the set of equations

$$x_1 = x_2 = x_3 = 0\,, \qquad x_4 - \eta x_5 = 0\,, \tag{1.5}$$

where η is a fifth root of unity. On the other hand, the D2 brane corresponds to

$$x_1 = 0\,, \qquad x_2 - \eta_1 x_3 = 0\,, \qquad x_4 - \eta_2 x_5 = 0\,. \tag{1.6}$$

As was also shown in [13] the corresponding factorisations do not correspond to any of the Recknagel-Schomerus (RS) boundary states [15]; indeed, it has been known for some time, that the charges of the RS boundary states only describe a sublattice of finite index in the complete charge lattice of the quintic theory [16].

Given our analysis of the correspondence for superpotentials of the form (1.4), we are able to identify the factorisations corresponding to (1.5) and (1.6) with specific permutation D-branes [17]. In particular, we find stable D0 branes at the Gepner point for a number of Calabi-Yau manifolds.[1] Furthermore we have checked that these boundary states have all the required properties: in particular, they carry the correct charges in the large volume basis (as can be confirmed by relating their charges to those of the RS branes), and for the case of the D0-brane they posses three complex marginal operators, corresponding to the motion of a single D0-brane on the Calabi-Yau manifold.

We have also noted that the intersection matrix of the D0-brane and its images under the Gepner monodromy always agrees with 'the intersection matrix in the Gepner basis' that had been computed for one- and two-parameter examples in [16, 19 – 21]. This basis was obtained by analytic continuation of the fundamental period at large volume [22 – 26]. In some examples, this basis already generates the full charge lattice; in other cases (including for example the quintic), a basis for the complete charge lattice is generated by the D2-brane and its images under the Gepner monodromy.

In general the equations (1.5) do not necessarily describe a point on the Calabi-Yau. In particular, if the point described by (1.5) lies on a singular locus, it is replaced by an exceptional set, and the corresponding brane is higher dimensional, a D2 or D4 brane depending on the geometry. In such situations one should expect that the relevant permutation brane carries the corresponding charge, as we verify in an explicit example.

In most of these examples, the RS branes describe a sublattice of maximal rank in the full charge lattice. In general, however, the RS branes do not account for all the charges; in particular, the RS branes sometimes do not carry the charges of branes that are wrapped around these exceptional divisors. As an example we consider a four parameter-model with two non-toric deformations, for which the RS branes only span a lower-dimensional sublattice of the charge lattice. In this model the form of the factorisation suggests that the permutation branes should carry the missing charges. This can be verified by showing that the charge lattice generated by the permutation branes has indeed full rank. On the

[1] For the case of the quintic, it was already realised in [18] that the relevant permutation boundary state only carries D0-brane charge.

other hand, the permutation branes do not always account for the full charge lattice as we also demonstrate with an example. In this case the failure of the permutation branes (and the tensor product branes) to generate the full charge lattice has a simple geometrical interpretation.

The paper is organised as follows. In section 2 we briefly review the analysis for the case of a single minimal model. Section 3 describes some of the matrix factorisations for the case of the tensor product of two minimal models, and studies their properties. For a restricted class of factorisations we propose corresponding boundary states in section 4, where a number of consistency checks are spelled out. The application to the construction of the D0-brane and D2 brane in Gepner models is given in section 5, and their geometric interpretation is discussed in section 6. There are various appendices where some of the more technical material is given.

Note added: while we were in the process of writing up this paper, we were made aware of related work [27]. After completion of this paper, the paper [28] appeared in which the relation between certain matrix factorisations and geometry is analysed using orbifold techniques.

2. The baby example: a single minimal model

Let us begin by briefly reviewing the correspondence for the case of a single minimal model [2−4] which corresponds to the superpotential

$$W = x^d \,. \tag{2.1}$$

The corresponding conformal field theory is described by a single $N = 2$ minimal model with $d = k + 2$. (Our conventions for the $N = 2$ minimal models are summarised in appendix A.) The spectrum of this theory is (after GSO-projection)

$$\mathcal{H} = \bigoplus_{[l,m,s]} \left(\mathcal{H}_{[l,m,s]} \otimes \bar{\mathcal{H}}_{[l,m,-s]}\right) . \tag{2.2}$$

This GSO-projection is the analogue of the Type 0A projection; there is also another GSO-projection (the analogue of Type 0B) for which the right-movers lie in the representation $[l, m, s]$ rather than $[l, m, -s]$. Since we think of embedding this model into a critical string theory we can always take either of the two GSO-projections for this internal theory, as long as we compensate this by taking the appropriate GSO-projection for the remaining degrees of freedom. As we shall see, the D-branes we are about to construct lie in (2.2). We are interested in B-type gluing conditions

$$\begin{aligned} \left(L_n - \bar{L}_{-n}\right) \|B\rangle\!\rangle &= 0 \\ \left(J_n + \bar{J}_{-n}\right) \|B\rangle\!\rangle &= 0 \\ \left(G^{\pm}_r + i\,\eta\, \bar{G}^{\pm}_{-r}\right) \|B\rangle\!\rangle &= 0 \,. \end{aligned} \tag{2.3}$$

Here $\eta = \pm 1$ describes the two spin-structures. The corresponding Ishibashi states are then supported in the sectors $[l, m, s] \otimes [l, -m, -s]$. For the above spectrum (2.2) we therefore have Ishibashi states $|[l,0,s]\rangle\rangle$ in each sector $[l,0,s]$ with $l+s$ even; in total there are therefore $2(k+1)$ Ishibashi states. (This discussion is appropriate for the bosonic subalgebra of the $N=2$ algebra; if we think in terms of the $N=2$ symmetry, then the two representations $[l,m,s]$ and $[l,m,s+2]$ form one $N=2$ representation. There are therefore only $k+1$ different $N=2$ representations, but we can choose the two different spin structures $\eta=\pm 1$ in each case, and therefore there are also $2(k+1)$ $N=2$ Ishibashi states. Half of them have $\eta=+1$, the other half $\eta=-1$.)

The corresponding B-type boundary states were constructed some time ago (see for example [29]), and are explicitly given as

$$\|L,S\rangle\rangle = \sqrt{k+2}\sum_{l+s\in 2\mathbb{Z}} \frac{S_{L0S,l0s}}{\sqrt{S_{l0s,000}}}\,|[l,0,s]\rangle\rangle\,. \tag{2.4}$$

Here $L=0,1,\ldots,k$ and $S=0,1,2,3$. The boundary states with S even (odd) satisfy the gluing conditions with $\eta=+1$ ($\eta=-1$); in the following we shall restrict ourselves to the case $\eta=+1$, and thus to even S.[2] We also note that

$$\|L,S\rangle\rangle = \|k-L,S+2\rangle\rangle \tag{2.5}$$

and thus there are only $k+1$ different boundary states with $\eta=+1$ (and $k+1$ different boundary states with $\eta=-1$). These boundary states therefore account for all the $N=2$ Ishibashi states. Finally we note that $\|L,S\rangle\rangle$ and $\|L,S+2\rangle\rangle$ are anti-branes of one another (since they differ by a sign in the coupling to the RR sector states). The corresponding open string spectrum can be determined from the overlap

$$\begin{aligned}\langle\langle LS\| q^{\frac{1}{2}(L_0+\bar{L}_0)-\frac{c}{24}}\,\|\,\hat{L}\hat{S}\rangle\rangle &= \\ &= \sum_{[l,m,s]}\Big(\delta^{(4)}(\hat{S}-S+s)\,N_{\hat{L}\ l}{}^{L} + \delta^{(4)}(\hat{S}-S+2+s)\,N_{\hat{L}\ k-l}{}^{L}\Big)\,\chi_{[l,m,s]}(\tilde{q})\,,\end{aligned} \tag{2.6}$$

where $N_{L\ l}{}^{M}$ denotes the level k fusion rules of $su(2)$, and $\chi_{[l,m,s]}$ is the character of the coset representation. In particular, we can read off from this expression how many topological chiral primary states propagate between two such branes; for example, between the two branes $\|L,0\rangle\rangle$ and $\|\hat{L},0\rangle\rangle$ we have as many chiral primary states $[l,l,0]$ as there are l for which $N_{\hat{L}\ l}^{L}=1$.

These results can now be compared with the analysis based on matrix factorisations [2, 4]. The corresponding factorisations of $W=x^d$ are

$$Q_r = \begin{pmatrix} 0 & x^r \\ x^{d-r} & 0 \end{pmatrix}\,, \qquad J = x^r\,, \qquad E = x^{d-r}\,, \tag{2.7}$$

where $r=1,2,\ldots,d-1$. (As was shown in [30], all factorisations of $W=x^d$ are equivalent to direct sums of these one-dimensional factorisations.) The dictionary between the two

[2] The D-branes corresponding to $\eta=-1$ or S odd preserve a different supercharge at the boundary. The branes that are described by the different matrix factorisations however always preserve the *same* supercharge at the boundary.

approaches is then

$$Q_r \Longleftrightarrow \| r-1, 0\rangle\!\rangle \,. \tag{2.8}$$

In particular, the two factorisations Q_r and Q_{d-r} that are related by interchanging the roles of E and J correspond to anti-branes of one another.

This relationship can be confirmed by comparing the topological open string spectra between two such branes. From the point of view of matrix factorisations this amounts to finding ϕ_0 and ϕ_1 such that the diagram

$$\begin{tikzcd}
Q & \mathbb{C}[x] \arrow[r, shift left, "J"] \arrow[d, "\phi_0"'] \arrow[dr, "t_0" near start] & \mathbb{C}[x] \arrow[l, shift left, "E"] \arrow[d, "\phi_1"] \arrow[dl, "t_1"' near start] \\
\hat{Q} & \mathbb{C}[x] \arrow[r, shift left, "\hat{J}"] & \mathbb{C}[x] \arrow[l, shift left, "\hat{E}"]
\end{tikzcd}$$

commutes. Here ϕ_0 and ϕ_1 are again polynomials in x, and the commutativity simply means that

$$\begin{aligned} 0 &= (D\phi)_0 = \hat{J}\,\phi_0 - \phi_1\,J\,, \\ 0 &= (D\phi)_1 = \hat{E}\,\phi_1 - \phi_0\,E\,. \end{aligned} \tag{2.9}$$

More abstractly, this is the condition that the morphism defined by

$$\Phi = \begin{pmatrix} \phi_0 & 0 \\ 0 & \phi_1 \end{pmatrix} \tag{2.10}$$

is Q-closed. In addition, Φ has to respect the U(1) grading (see [31, 13] for a detailed discussion of this point); this is trivial for the case in question, but in general imposes a non-trivial constraint. The actual topological states are described by the Q-cohomology [1–4, 31]; thus we need to determine the solutions (ϕ_0, ϕ_1) up to Q-exact solutions. In the current context the Q-exact solutions are

$$\begin{aligned} \tilde{\phi}_0 &= (Dt)_0 = \hat{E}\,t_0 + t_1\,J\,, \\ \tilde{\phi}_1 &= (Dt)_1 = \hat{J}\,t_1 + t_0\,E\,. \end{aligned} \tag{2.11}$$

The Q-cohomology of (ϕ_0, ϕ_1) describes then the 'bosonic' open string degrees of freedom, i.e. the topological open string states between the branes corresponding to Q and $\hat{Q}$. The 'fermionic' degrees of freedom, i.e. the topological open string states between the brane Q and the anti-brane of $\hat{Q}$, can be deduced from this by exchanging the roles of $\hat{E}$ and $\hat{J}$. These degrees of freedom are then described by (t_0, t_1). Their Q-closedness condition is

$$\begin{aligned} 0 &= (Dt)_0 = \hat{E}\,t_0 + t_1\,J \\ 0 &= (Dt)_1 = \hat{J}\,t_1 + t_0\,E\,, \end{aligned} \tag{2.12}$$

and the Q-exact states are those that are of the form

$$\begin{aligned}\tilde{t}_0 &= (D\phi)_0 = \hat{J}\,\phi_0 - \phi_1\,J\,,\\ \tilde{t}_1 &= (D\phi)_1 = \hat{E}\,\phi_1 - \phi_0\,E\,.\end{aligned} \tag{2.13}$$

For example, for $Q = \hat{Q} = Q_r$, the Q-closed condition for the bosonic degrees of freedom is simply $\phi_0 = \phi_1$. The Q-exact solutions are those for which $\phi_0 = \phi_1$ contains x^r (or x^{d-r}) as a factor. Thus for $r \leq (d-1)/2$ we have r different topological states, corresponding to $\phi_0 = \phi_1 = 1, x, \ldots, x^{r-1}$. This agrees then precisely with the topological open string spectrum of $\|r-1, 0\rangle\!\rangle$ where the chiral primaries $[l, l, 0]$ with $l = 0, 2, \ldots, 2(r-1)$ appear. The analysis for other combinations of branes works likewise. One can also check that the U(1)-charges match.

3. The product theory

Now we want to consider the product theory that corresponds to the superpotential

$$W = x_1^d + x_2^d\,. \tag{3.1}$$

The space of states of the corresponding conformal field theory is (after GSO-projection)

$$\begin{aligned}\mathcal{H} = \bigoplus_{[l_1,m_1,s_1],[l_2,m_2,s_2]} &\Big((\mathcal{H}_{[l_1,m_1,s_1]} \otimes \mathcal{H}_{[l_2,m_2,s_2]}) \otimes (\bar{\mathcal{H}}_{[l_1,m_1,s_1]} \otimes \bar{\mathcal{H}}_{[l_2,m_2,s_2]}) \oplus \\ &\oplus (\mathcal{H}_{[l_1,m_1,s_1]} \otimes \mathcal{H}_{[l_2,m_2,s_2]}) \otimes (\bar{\mathcal{H}}_{[l_1,m_1,s_1+2]} \otimes \bar{\mathcal{H}}_{[l_2,m_2,s_2+2]}) \Big)\,,\end{aligned} \tag{3.2}$$

where the sums over s_1 and s_2 are restricted to $s_1 - s_2 \in 2\mathbb{Z}$. (Again, there is also another GSO-projection, but as we shall see the branes we are about to construct will lie in (3.2).)

This theory has the obvious tensor product branes that satisfy the gluing conditions (2.3) separately for the two $N = 2$ theories; they are explicitly given by[3]

$$\begin{aligned}\|L_1, S_1, L_2, S_2\rangle\!\rangle = \frac{(2k+4)}{4\sqrt{2}} \sum_{s_1,s_2} \sum_{l_1,l_2} &\left(\frac{S_{L_1 0 S_1, l_1 0 s_1}}{\sqrt{S_{000,l_1 0 s_1}}} + \frac{S_{k-L_1 0 S_1+2, l_1 0 s_1}}{\sqrt{S_{000,l_1 0 s_1}}} \right) \times \\ \times &\left(\frac{S_{L_2 0 S_2, l_2 0 s_2}}{\sqrt{S_{000,l_2 0 s_2}}} + \frac{S_{k-L_2 0 S_2+2, l_2 0 s_2}}{\sqrt{S_{000,l_2 0 s_2}}} \right) |[l_1, 0, s_1] \otimes [l_2, 0, s_2]\rangle\!\rangle\,,\end{aligned}$$

where $|[l_1, 0, s_1] \otimes [l_2, 0, s_2]\rangle\!\rangle$ is now the Ishibashi state in the sector

$$|[l_1, 0, s_1] \otimes [l_2, 0, s_2]\rangle\!\rangle \in \Big(\mathcal{H}_{[l_1,0,s_1]} \otimes \mathcal{H}_{[l_2,0,s_2]}\Big) \otimes \Big(\bar{\mathcal{H}}_{[l_1,0,-s_1]} \otimes \bar{\mathcal{H}}_{[l_2,0,-s_2]}\Big)\,, \tag{3.3}$$

and the sum over l_i and s_i is unrestricted, except that $s_1 - s_2$ is even. As before S_i even (odd) describes the boundary states that satisfy the gluing condition for the ith $N = 2$ algebra with $\eta = +1$ ($\eta = -1$); we shall therefore restrict ourselves to considering S_1 and S_2 even. Furthermore, the labels (L_i, S_i) are only defined up to the equivalence

[3]We are ignoring in the following the resolved branes that appear for $L_1 = L_2 = k/2$ if k is even.

$(L_i, S_i) \sim (k - L_i, S_i + 2)$ and $(S_1, S_2) \sim (S_1 + 2, S_2 + 2)$. The open string spectrum between two such branes is essentially given by the tensor product of (2.6)

$$
\begin{aligned}
&\langle\langle L_1, S_1, L_2, S_2 \| q^{\frac{1}{2}(L_0+\bar{L}_0)-\frac{c}{12}} \| \hat{L}_1, \hat{S}_1, \hat{L}_2, \hat{S}_2]\rangle\rangle = \\
&= \sum_{r=0}^{1} \sum_{[l_1 m_1 s_1],[l_2 m_2 s_2]} \Big(\delta^{(4)}(\hat{S}_1 - S_1 + s_1 + 2r)\, N_{\hat{L}_1 l_1}{}^{L_1} + \\
&\qquad\qquad + \delta^{(4)}(\hat{S}_1 + 2 - S_1 + s_1 + 2r)\, N_{k-\hat{L}_1 l_1}{}^{L_1} \Big) \times \\
&\qquad\qquad \times \Big(\delta^{(4)}(\hat{S}_2 - S_2 + s_2 - 2r)\, N_{\hat{L}_2 l_2}{}^{L_2} + \\
&\qquad\qquad + \delta^{(4)}(\hat{S}_2 + 2 - S_2 + s_2 - 2r)\, N_{k-\hat{L}_2 l_2}{}^{L_2} \Big) \times \\
&\qquad\qquad \times \chi_{[l_1,m_1,s_1]}(\tilde{q})\, \chi_{[l_2,m_2,s_2]}(\tilde{q}) \,.
\end{aligned}
$$

As before, (L_1, S_1, L_2, S_2) and $(L_1, S_1 + 2, L_2, S_2)$ are anti-branes of one another. For future reference we also note that these tensor product branes do *not* couple to any RR ground states; they therefore do not carry any RR charge. These boundary states correspond to tensor products of the one-dimensional factorisations we considered before [13, 31]. More specifically, the boundary state $\|L_1, 0, L_2, 0\rangle\rangle$ corresponds to the factorisation

$$
\mathbb{C}[x_1, x_2]^{\oplus 2} \underset{E}{\overset{J}{\rightleftarrows}} \mathbb{C}[x_1, x_2]^{\oplus 2} \;, \qquad \text{with} \qquad J = \begin{pmatrix} J_2 & J_1 \\ E_1 & -E_2 \end{pmatrix} \,, \qquad E = \begin{pmatrix} E_2 & J_1 \\ E_1 & -J_2 \end{pmatrix} \tag{3.4}
$$

and $J_i = x_i^{L_i+1}$ and $E_i = x_i^{d-1-L_i}$, such that $W = E_1 J_1 + E_2 J_2$. This follows essentially from the same analysis as for the case of a single minimal model.

In addition to these tensor product factorisations, there are however also rank 1 factorisations [13]. The corresponding boundary conditions had been studied previously using Landau-Ginzburg techniques in [32]. These factorisations make use of the fact that the superpotential can be factorised as

$$
W = \prod_{\eta} (x_1 - \eta x_2) \,, \tag{3.5}
$$

where η is in turn each of the d different d'th roots of -1. We can label these roots as

$$
\eta_m = e^{-\pi i \frac{2m+1}{d}} \,, \tag{3.6}
$$

where $m = 0, 1, \ldots, d-1$. Let us define $D = \{0, \ldots, d-1\}$. Then we can construct the rank 1 factorisations

$$
\mathbb{C}[x_1, x_2] \underset{E}{\overset{J}{\rightleftarrows}} \mathbb{C}[x_1, x_2] \;, \tag{3.7}
$$

where

$$
J = \prod_{m \in I} (x_1 - \eta_m x_2) \,, \qquad E = \prod_{n \in D \setminus I} (x_1 - \eta_n x_2) \,. \tag{3.8}
$$

Here I is any subset of D.

At this stage it is not clear what $N = 2$ boundary states these factorisations correspond to (we shall make a proposal for at least some of these factorisations in section 4). In order to be able to make an identification, we should obtain as much information about them as possible. In particular we need to determine the open string spectra involving these rank 1 factorisations. As we shall see, the result will have a simple geometric interpretation, so it may be helpful to review first the relation between matrix factorisations and geometry.

3.1 Matrix factorisations and geometry

In [33] Orlov showed that the category of D-branes in Landau Ginzburg theories is equivalent to a certain geometrical category $D_{Sg}(X)$ that is non-trivial only on singular varieties X. To be more precise, topological B-type D-branes on a variety X correspond geometrically to the bounded derived category of coherent sheaves on X. On a smooth variety, any such sheaf has a finite locally free resolution. This is no longer the case on a singular variety, which was the motivation in [33] to introduce $D_{Sg}(X)$ as the quotient of the bounded derived category of coherent sheaves by the subcategory of finite complexes of locally free sheaves. To understand the relationship with Landau-Ginzburg models, in particular in our example, we start from a Landau Ginzburg potential $W : \mathbb{C}^n \to \mathbb{C}$ with an isolated critical point at the origin, in our case, $W = x_1^d + x_2^d : \mathbb{C}^2 \to \mathbb{C}$. Denoting the fiber of W over 0 by S_0, [33] allows us to establish a relation between $D_{Sg}(S_0)$ and the Landau-Ginzburg category. For this, we associate with any factorisation

$$\left(P_1 \underset{E}{\overset{J}{\rightleftarrows}} P_0 \right)$$

the short exact sequence

$$0 \longrightarrow P_1 \xrightarrow{J} P_0 \longrightarrow \text{Coker } J \longrightarrow 0\,. \tag{3.9}$$

The geometrical object associated to the factorisation is then the sheaf Coker J, which, since it is annihilated by W, is a sheaf on S_0. Let us apply this to the simplest rank 1 factorisation, where J is a single linear factor $J = x_1 - \eta x_2$. Coker J is then simply the ring $\mathbb{C}[x_1, x_2]/J$, or, geometrically, the line with the equation

$$x_1 - \eta x_2 = 0\,.$$

For higher order J, $J = \prod_{m\in I}(x_1 - \eta_m x_2)$, we obtain accordingly a union of lines

$$\bigcup_{m\in I} \{x_1 - \eta_m x_2 = 0\}\,.$$

Based on this geometric picture, we can make a prediction for the number of fermionic (and bosonic) operators between pairs of branes. The idea is simply that the number of fermions corresponds to the number of intersections between the two branes. For example, on a single brane, there should be no fermions, whereas between two branes corresponding to different single lines (J linear) there should be exactly one fermion as calculated in [13]. According to this logic, between disjoint sets of lines there should be $d_1 d_2$ fermions, where d_1 is the number of lines in the first set, and d_2 the number of lines in the second set.

We can also count the bosons, since by definition the number of bosons propagating between two branes is equal to the number of fermions propagating between the brane and the anti-brane. Exchanging brane and anti-brane corresponds to exchanging J and E in Landau-Ginzburg language, which means, in this geometrical language, that the anti-brane of a brane localised at a given set of lines consists of the complementary set of lines. Therefore, the number of bosons is again given by an intersection number, this time between the branes and anti-branes.

Finally, the tensor product branes can be thought of as corresponding to points at the origin, where the number of points is determined by the factorisation labels L_1, L_2. In particular, they are invariant under rotations. This corresponds to the fact that these branes do not carry RR charge.

We will now compute the open string spectrum from the matrix factorisation point of view and check that the dimensions of the cohomologies indeed match these geometric expectations.

3.2 Calculating the open string spectrum

First we consider the open string spectrum between two rank 1 branes. As a warm-up we consider the case of open strings between a brane and itself. For the fermions the BRST-invariance condition is

$$E\, t_0 + t_1\, J = 0\,. \tag{3.10}$$

Since E and J do not have any common divisors, the only solution is $t_1 = aE$ and $t_0 = -aJ$. It is then easy to see that this solution is BRST trivial, and thus there are no fermions on a single rank 1 brane.
For the bosons, the BRST-invariance condition is

$$J\, \phi_0 = \phi_1\, J\,, \qquad E\, \phi_1 = \phi_0\, E\,, \tag{3.11}$$

from which it can be concluded that $\phi_0 = \phi_1$. The boson is BRST trivial if

$$\phi_0 = E\, t_0 + t_1\, J\,, \qquad \phi_1 = J\, t_1 + t_0\, E\,, \tag{3.12}$$

which means that the bosonic spectrum is given by the ring $\mathbb{C}[x_1, x_2]/I$, where I is the ideal generated by J and E. To calculate the dimension of this space, we note that the dimension corresponds to the number of intersections of the two curves $E = 0$ and $J = 0$. According to Bezouts theorem, two plane curves of degrees d_0 and d_1 will intersect in $d_1 d_0$ points, counting intersections at infinity and multiplicities. Since in our case there are no intersections at infinity, the dimension of the ideal is given by $d_0 d_1$. This is therefore in perfect agreement with the geometric picture outlined above.

The general case can be shown along similar lines; the details of this calculation are described in appendix B. There we also give the calculation for the open string spectrum between a tensor product and a rank 1 brane (for the case that the latter has $\hat{J}$ linear).

3.3 Matrix flows

The above picture also suggests that the branes corresponding to higher order J can be obtained as bound states of the branes with linear J. We want to explain now that this is actually correct.

As was explained in [30] there is a natural notion of isomorphism of matrix factorisations: two matrix factorisations Q_1 and Q_2 are isomorphic if $Q_2 = UQ_1U^{-1}$, where both U and its inverse U^{-1} are block-diagonal matrices

$$U = \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix} \tag{3.13}$$

with polynomial entries. In particular, this condition implies that the spectrum of Q_1 and Q_2 relative to any other brane is the same; thus it is indeed natural to identify such factorisations. This concept is crucial to understand the bound state formation of branes from the topological point of view [34]. Suppose now that we are given a pair of branes

$$\Big(P_0 \underset{E_P}{\overset{J_P}{\rightleftarrows}} P_1 \Big) \qquad \text{and} \qquad \Big(O_0 \underset{E_O}{\overset{J_O}{\rightleftarrows}} O_1 \Big), \tag{3.14}$$

whose relative open string contains a tachyon. By this we mean a boundary changing operator $t = (t_0, t_1)$, $t_0 : P_0 \to O_1$, $t_1 : P_1 \to O_0$ that is BRST closed but not BRST exact. Likewise, there can be tachyons in the other direction $t' = (t'_0, t'_1)$, $t'_0 : O_0 \to P_1$ and $t'_1 : O_1 \to P_0$. A bound state of these two branes with that tachyon profile should have the form

$$\Big(P_0 \oplus O_0 \underset{E}{\overset{J}{\rightleftarrows}} P_1 \oplus O_1 \Big), \tag{3.15}$$

where the maps J and E are given by

$$J = \begin{pmatrix} J_P & t'_0 \\ t_0 & J_O \end{pmatrix}, \qquad E = \begin{pmatrix} E_P & t'_1 \\ t_1 & E_O \end{pmatrix}. \tag{3.16}$$

The BRST operator of the combined system is then

$$Q = \begin{pmatrix} 0 & J \\ E & 0 \end{pmatrix}. \tag{3.17}$$

The condition that a bound state is formed in this way is that (3.15) is a valid boundary condition fulfilling $Q^2 = W$. Using the above notion of isomorphism, the resulting boundary condition may in fact be equivalent to one of the other boundary conditions. This occurs in particular for the case where we have two rank 1 branes with complementary factors.

3.3.1 Rank 1 flows

Let us consider the superposition of two rank 1 branes

$$J_P = \prod_{n \in I_P} (x_1 - \eta_n x_2), \qquad E_P = \prod_{n' \in D \setminus I_P} (x_1 - \eta_{n'} x_2), \tag{3.18}$$

and

$$J_O = \prod_{m \in I_O} (x_1 - \eta_m x_2)\,, \qquad E_O = \prod_{m' \in D \backslash I_O} (x_1 - \eta_{m'} x_2)\,, \tag{3.19}$$

where I_P and I_O are disjoint, $I_P \cap I_O = \emptyset$. Then E_P contains J_O as a factor, and E_O contains J_P as a factor. As we have explained before, the open string spectrum of this configuration contains $|I_P|\,|I_O|$ fermions. We can therefore modify the BRST operator of this configuration by

$$Q = \begin{pmatrix} 0 & 0 & J_P & 0 \\ 0 & 0 & \lambda & J_O \\ E_P & 0 & 0 & 0 \\ -\lambda \tilde{E} & E_O & 0 & 0 \end{pmatrix} \qquad \tilde{E} = \frac{E_P}{J_O} = \frac{E_O}{J_P}\,, \tag{3.20}$$

where λ labels the different fermionic perturbations. One easily checks that this Q still squares to the superpotential. For constant λ one then finds that this BRST operator is equivalent to the BRST operator

$$\hat{Q} = \begin{pmatrix} 0 & 0 & 0 & J_P\,J_O \\ 0 & 0 & 1 & 0 \\ 0 & W & 0 & 0 \\ \tilde{E} & 0 & 0 & 0 \end{pmatrix}. \tag{3.21}$$

In fact, the relevant invertible matrix U that satisfies $Q\,U = U\,\hat{Q}$ is given by

$$U = \begin{pmatrix} a & J_P\,d & 0 & 0 \\ 0 & \lambda\,d & 0 & 0 \\ 0 & 0 & d & a\,J_O \\ 0 & 0 & 0 & -\lambda\,a \end{pmatrix}. \tag{3.22}$$

It is clear that the inverse of this matrix (for constant a, d, λ) is again a matrix with polynomial entries. On the other hand, the BRST operator $\hat{Q}$ simply describes the superposition of a trivial brane (corresponding to the trivial matrix factorisation) with the rank 1 brane described by

$$\tilde{J} = J_P\,J_O = \prod_{n \in I_P \cup I_O} (x_1 - \eta_n x_2)\,, \qquad \tilde{E} = \prod_{n' \in D \backslash \{I_P \cup I_O\}} (x_1 - \eta_{n'} x_2)\,. \tag{3.23}$$

This argument therefore shows that two complementary rank 1 branes can flow to the rank 1 brane that is described by their product. This thus confirms the geometric picture that was put forward in section 3.1.

3.3.2 Flows from rank 1 to tensor product branes

The other flow that is of interest relates rank 1 branes to tensor product branes. The simplest example concerns the configuration of a rank 1 brane (corresponding to a single factor) with its anti-brane, i.e.

$$J_P = (x_1 - \eta_n x_2)\,, \qquad E_P = \prod_{n' \neq n} (x_1 - \eta_{n'} x_2)\,, \tag{3.24}$$

and

$$J_O = \prod_{m\neq n} (x_1 - \eta_m x_2)\,, \qquad E_O = (x_1 - \eta_n x_2)\,. \tag{3.25}$$

This is a special case of the situation considered in the previous subsection, and the BRST operator is therefore again of the form (3.20) with $\tilde{E} = 1$. If we take λ to be constant, then the above analysis implies that the configuration flows to the trivial brane configuration (since $J_P J_O = W$). In order to flow to a non-trivial brane configuration, we therefore consider

$$\lambda = x_1 + \eta_n\, x_2\,. \tag{3.26}$$

Then one finds that this BRST matrix is equivalent to

$$\hat{Q} = \begin{pmatrix} \mathbf{0} & r_1 \\ r_0 & \mathbf{0} \end{pmatrix}, \qquad r_1 = \begin{pmatrix} x_2 & x_1^{d-1} \\ x_1 & -x_2^{d-1} \end{pmatrix}, \qquad r_0 = \begin{pmatrix} x_2^{d-1} & x_1^{d-1} \\ x_1 & -x_2 \end{pmatrix}, \tag{3.27}$$

where the relevant U-matrix satisfying $Q\,U = U\,\hat{Q}$ is simply

$$U = \begin{pmatrix} -\eta_n\, d & d & 0 & 0 \\ \eta_n\, d & d & 0 & 0 \\ 0 & 0 & d & -d\,v \\ 0 & 0 & 0 & 2\eta_n\, d \end{pmatrix}, \qquad v = \frac{x_2^{d-1} + \eta_n x_1^{d-1}}{x_1 - \eta_n x_2}\,. \tag{3.28}$$

In particular, one observes that v is a polynomial in x_1 and x_2 (since η_n is a dth root of -1), and therefore U, as well as its inverse, are matrices with polynomial entries.

On the other hand, we recognise the BRST matrix (3.27) to be the tensor product brane corresponding to

$$J_1 = x_1^{d-1}\,, \qquad J_2 = x_2\,. \tag{3.29}$$

Repeating this construction we can therefore obtain any of the tensor product branes from the rank 1 branes. To this end we observe, following [34], that there is a flow relating the tensor product branes

$$J_1 = x_1^m\,,\, J_2 = x_2 \quad \oplus \quad \tilde{J}_1 = x_1^m\,,\, \tilde{J}_2 = x_2^l \qquad \longrightarrow \qquad \hat{J}_1 = x_1^m\,,\, \hat{J}_2 = x_2^{l+1}\,, \tag{3.30}$$

and similarly

$$J_1 = x_1^{d-1}\,,\, J_2 = x_2^n \;\oplus\; \tilde{J}_1 = x_1^{d-l}\,,\, \tilde{J}_2 = x_2^n \quad \longrightarrow \quad \hat{J}_1 = x_1^{d-l-1}\,,\, \hat{J}_2 = x_2^n\,. \tag{3.31}$$

Combining these two flows, it is then clear that every tensor product brane can be obtained from a suitable combination of the tensor product brane that is described by (3.29). In turn, this last brane could be obtained from two rank 1 branes. Combining these arguments, it therefore follows that *every* tensor product brane can be obtained from a suitable combination of the rank 1 branes.

4. Permutation branes

In the previous section we have analysed the properties of the rank 1 factorisations in detail. Now we want to make a proposal for which $N = 2$ superconformal boundary states these factorisations correspond to.

The diagonal $N = 2$ algebra (which is the symmetry that is relevant from the matrix factorisation point of view) has central charge $2c$; except for the case of $k = 1$ that was discussed in [35], the diagonal $N = 2$ algebra therefore does not define a minimal model. Regarded as a theory with respect to this algebra, the theory is therefore (for $k > 1$) not rational. It is then difficult to find all $N = 2$ boundary states of this theory.[4] However, there are always two classes of 'rational' D-branes one can easily construct: the tensor product branes we have considered at the beginning of section 3, and the permutation branes [17] (see also [37]). The permutation branes are characterised by the gluing conditions

$$\begin{aligned}\left(L_n^{(1)} - \bar{L}_{-n}^{(2)}\right) \|B\rangle\rangle = \left(L_n^{(2)} - \bar{L}_{-n}^{(1)}\right) \|B\rangle\rangle &= 0 \\ \left(J_n^{(1)} + \bar{J}_{-n}^{(2)}\right) \|B\rangle\rangle = \left(J_n^{(2)} + \bar{J}_{-n}^{(1)}\right) \|B\rangle\rangle &= 0 \\ \left(G_r^{\pm(1)} + i\eta_1 \bar{G}_{-r}^{\pm(2)}\right) \|B\rangle\rangle = \left(G_r^{\pm(2)} + i\eta_2 \bar{G}_{-r}^{\pm(1)}\right) \|B\rangle\rangle &= 0\,. \end{aligned} \tag{4.1}$$

Provided that $\eta_1 = \eta_2$, these gluing conditions imply that the diagonal $N = 2$ gluing conditions are respected. For the theory under consideration (3.2) we have permutation Ishibashi states in the sectors

$$|[l, m, s_1] \otimes [l, -m, -s_2]\rangle\rangle^\sigma \in \left(\mathcal{H}_{[l,m,s_1]} \otimes \mathcal{H}_{[l,-m,-s_2]}\right) \otimes \left(\bar{\mathcal{H}}_{[l,m,s_2]} \otimes \bar{\mathcal{H}}_{[l,-m,-s_1]}\right). \tag{4.2}$$

The corresponding boundary states are then

$$\begin{aligned}&\| \; [\; L, M, S_1, S_2 \;] \; \rangle\rangle = \\ &\quad = \frac{1}{2\sqrt{2}} \sum_{l,m,s_1,s_2} \frac{S_{Ll}}{S_{0l}} \, e^{i\pi M m/(k+2)} \, e^{-i\pi(S_1 s_1 - S_2 s_2)/2} \, |[l, m, s_1] \otimes [l, -m, -s_2]\rangle\rangle^\sigma \,,\end{aligned} \tag{4.3}$$

where the sum runs over all l, m, s_1 and s_2 for which

$$l + m + s_1 \qquad \text{and} \qquad s_1 - s_2 \quad \text{are even}\,. \tag{4.4}$$

The labels $[L, M, S_1, S_2]$ are defined for $L + M + S_1 - S_2$ even only. Again, S_1 and S_2 correspond to the choice of the spin structures η_1 and η_2, respectively; in order to preserve the diagonal $N = 2$ algebra we therefore need that $S_1 - S_2$ is even, in which case also $L+M$ is even. As before we shall only consider the case that $\eta_1 = \eta_2 = +1$, i.e. that both S_1 and S_2 are even. Note that the boundary state is invariant under replacing both S_1 and S_2 by $S_i \mapsto S_i + 2$. Furthermore we have the equivalence $[L, M, S_1, S_2] \simeq [k - L, M + k + 2, S_1 + 2, S_2]$. The branes with (S_1, S_2) and $(S_1 + 2, S_2)$ are anti-branes of one another.

[4] For the case $k = 2$ for which $2c = 3$, the techniques of [36] should allow one to find a complete description of these boundary states; this will be described elsewhere.

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In contradistinction to the tensor product branes, these permutation branes now couple to the RR ground states. In fact, the coefficient of the RR ground states in the sector

$$\left(\mathcal{H}_{[l,l+1,1]} \otimes \mathcal{H}_{[l,-l-1,-1]}\right) \otimes \left(\bar{\mathcal{H}}_{[l,l+1,1]} \otimes \bar{\mathcal{H}}_{[l,-l-1,-1]}\right) \tag{4.5}$$

in the boundary state labelled by $[L, M, S_1, S_2]$ is precisely

$$Q_l\left(\|[L, M, S_1, S_2]\rangle\!\rangle\right) = \frac{1}{\sqrt{2}}\, e^{i\pi S_2/2}\, \frac{S_{LMS_1,l(l+1)1}}{S_{000,l(l+1)1}}\,. \tag{4.6}$$

For the following it is also useful to understand the behaviour of these permutation branes under the $\mathbb{Z}_{k+2}$ axial symmetry. Recall that each minimal model has a $\mathbb{Z}_{k+2}$ symmetry, whose generator g acts on the states in $\mathcal{H}_{[l,m,s]} \otimes \bar{\mathcal{H}}_{[l,m,s']}$ as

$$g|_{\mathcal{H}_{[l,m,s]}\otimes\bar{\mathcal{H}}_{[l,m,s']}} = \exp\left(2\pi i \frac{m}{k+2}\right). \tag{4.7}$$

(Thus g acts as the simple current $[0, 2, 0]$.) It is easy to see from the explicit formula (4.3) that

$$g_1\, \|[L, M, S_1, S_2]\rangle\!\rangle = \|[L, M+2, S_1, S_2]\rangle\!\rangle\,, \quad g_2\, \|[L, M, S_1, S_2]\rangle\!\rangle = \|[L, M-2, S_1, S_2]\rangle\!\rangle\,. \tag{4.8}$$

In particular, the permutation boundary state is therefore invariant under $g_1\, g_2$.

4.1 The dictionary

We are now in the position to identify a subset of the rank 1 factorisations with permutation branes. The precise correspondence is as follows:

$$\|[L, M, S_1 = 0, S_2 = 0]\rangle\!\rangle \qquad \Longleftrightarrow \qquad J = \prod_{m=(M-L)/2}^{(M+L)/2} (x_1 - \eta_m x_2)\,. \tag{4.9}$$

For $\|[L, M, S_1 = 2, S_2 = 0]\rangle\!\rangle = \|[L, M, S_1 = 0, S_2 = 2]\rangle\!\rangle$ the roles of J and E are interchanged. We should note that this identifies only a subset of the rank 1 factorisations with permutation branes (since the phases that appear on the right hand side are 'consecutive'); it would be very interesting to understand how to describe the remaining factorisations in terms of conformal field theory. On the other hand, our proposal does account for all rank 1 factorisations with J linear — these are precisely the permutation branes with $L = 0$. Since these factorisations generate (upon forming bound states) all the factorisations we have considered, we are at least accounting for all the RR charges. Also, as we shall see, these are the factorisations that are relevant for the construction of the D0 and D2-brane in Gepner models.

Before we begin checking this proposal in some detail, it is useful to observe that it transforms at least correctly under the axial symmetries. The $\mathbb{Z}_{k+2}$ symmetry g_i in conformal field theory corresponds, in terms of matrix factorisations, to the maps

$$g_i: \qquad x_i \mapsto e^{\frac{2\pi i}{k+2}}\, x_i\,. \tag{4.10}$$

Under g_i, each factor $(x_1 - \eta_n x_2)$ therefore gets mapped to

$$g_1(x_1 - \eta_n x_2) = e^{\frac{2\pi i}{k+2}}(x_1 - \eta_{n+1} x_2) \tag{4.11}$$

and

$$g_2(x_1 - \eta_n x_2) = (x_1 - \eta_{n-1} x_2)\,, \tag{4.12}$$

respectively. Since overall factors do not matter, we therefore see that g_1 shifts $M \mapsto M+2$ on the right-hand-side of (4.9), while g_2 acts as $M \mapsto M-2$. This is then precisely in accord with the transformation properties of the permutation branes of (4.8).

There are three additional sets of consistency checks for this identification that we have performed; they will now be described in turn.

4.2 Open string spectrum between permutation branes

It is straightforward to calculate the corresponding open string spectrum, and one finds

$$\begin{aligned}
&\langle\langle [L,M,S_1,S_2] \| q^{\frac{1}{2}(L_0+\bar{L}_0)-\frac{c}{12}} \; \| \; [\hat{L},\hat{M},\hat{S}_1,\hat{S}_2] \rangle\rangle = \\
&= \sum_{[l'_i,m'_i,s'_i]} \chi_{[l'_1,m'_1,s'_1]}(\tilde{q})\, \chi_{[l'_2,m'_2,s'_2]}(\tilde{q}) \times \\
&\quad \times \sum_{\hat{l}} \Big[N_{\hat{l}\hat{L}}{}^{L}\, N_{l'_1 l'_2}{}^{\hat{l}}\, \delta^{(2k+4)}(\Delta M + m'_1 - m'_2) \times \\
&\qquad \times \Big(\delta^{(4)}(\Delta S_1 + s'_1)\, \delta^{(4)}(\Delta S_2 + s'_2) + \delta^{(4)}(\Delta S_1 + 2 + s'_1)\, \delta^{(4)}(\Delta S_2 + 2 + s'_2) \Big) + \\
&\qquad + N_{\hat{l}\, k-\hat{L}}{}^{L}\, N_{l'_1 l'_2}{}^{\hat{l}}\, \delta^{(2k+4)}(\Delta M + k + 2 + m'_1 - m'_2) \times \\
&\qquad \times \Big(\delta^{(4)}(\Delta S_1 + 2 + s'_1)\, \delta^{(4)}(\Delta S_2 + s'_2) + \delta^{(4)}(\Delta S_1 + s'_1)\, \delta^{(4)}(\Delta S_2 + 2 + s'_2) \Big) \Big]\,,
\end{aligned}$$

where $\Delta M = \hat{M} - M$ and $\Delta S_i = \hat{S}_i - S_i$.

Let us first consider the case $\Delta_i S = 0$, i.e. the overlap between branes and branes. (In the language of matrix factorisations these are the 'bosons'.) Since the sum runs over equivalence classes $[l'_i, m'_i, s'_i]$ and only even values of s'_i contribute, we may restrict ourselves, without loss of generality, to $s'_1 = s'_2 = 0$. The chiral primaries are then characterised by $l'_i = m'_i$. Since $m'_1 - m'_2 = M_1 - M_2$, and l'_2 is contained in the fusion product of L_1, L_2 and l'_1, we therefore get one topological chiral primary for each l'_1 and l'_2 for which

$$\textbf{boson}: \qquad l'_2 = l'_1 \pm (M_1 - M_2) \mod (2k+4) \qquad \text{and} \qquad l'_2 \subset L_1 \otimes L_2 \otimes l'_1\,. \tag{4.13}$$

In order to determine the topological 'fermions', we need to consider the overlap between a brane and an anti-brane, i.e. $\Delta S_1 = 2$ and $\Delta S_2 = 0$, say. Then an identical analysis leads to

$$\textbf{fermion}: \qquad l'_2 = -l'_1 - 2 \pm (M_1 - M_2) \mod (2k+4) \qquad \text{and} \qquad l'_2 \subset L_1 \otimes L_2 \otimes l'_1\,. \tag{4.14}$$

It follows from the first equation that there are no bosons if $L_1 + L_2 < |M_1 - M_2|$. (Here, as in the following, we shall assume that the M_i have been chosen such that $|M_1 - M_2| \leq k+2$.) On the other hand, if $L_1 + L_2 = |M_1 - M_2|$, then there are precisely $k + 1 - (L_1 + L_2)$

bosons; to see this one observes that only the representation L_1+L_2 in the fusion product of L_1 and L_2 can contribute in (4.13), and that one then gets one solution each for $l_1' = 0,1,2,\ldots,k-(L_1+L_2)$. If $L_1+L_2 = |M_1-M_2|-2$ we have in addition also $k+3-(L_1+L_2)$ solutions coming from the L_1+L_2-2 term in the fusion product of L_1 and L_2, leading to $k+1-(L_1+L_2)+k+3-(L_1+L_2)$ boson states. In the general case when $L_1+L_2 = |M_1-M_2|-2U$ we therefore have

$$\sum_{d=0}^{U}(k+1-(L_1+L_2)+2d) = (U+1)\,(k+1+U-(L_1+L_2)) \tag{4.15}$$

topological bosonic states. The number of fermions is given by the same formulae, except that we have to replace L_1 by $(k-L_1)$.

These results now need to be compared with the results of the calculation in appendix B, in particular, (B.11) and (B.7). Using the above identification (4.9), L_1 and L_2 are related to I and $\hat{I}$ as

$$L_1 = |I|-1\,, \qquad L_2 = |\hat{I}|-1\,. \tag{4.16}$$

Furthermore, $U<0$ corresponds precisely to the case $|I\cap\hat{I}| = \emptyset$. If this is the case, the number of topological boson states vanishes, in agreement with (B.11). On the other hand, if $U\geq 0$ we have the relation

$$U = |I\cap\hat{I}|-1\,. \tag{4.17}$$

Then (4.15) becomes

$$|\text{bosons}| = |I\cap\hat{I}|\,|D\setminus\{I\cup\hat{I}\}|\,, \tag{4.18}$$

and therefore agrees precisely with (B.11). The number of topological fermions can be obtained from either description upon replacing L_1 by $k-L_1$, and their number therefore also agrees.

4.3 Open string spectrum between permutation and tensor branes

The calculation of the open string spectrum between a permutation and a tensor product brane[5] is actually quite subtle. The subtle point concerns the calculation of the overlap between the tensor product Ishibashi state $|[l,0,s]\otimes[l,0,s]\rangle\!\rangle$ and the permutation Ishibashi state in the same sector, $|[l,0,s]\otimes[l,0,s]\rangle\!\rangle^{\sigma}$. On general grounds one knows that this overlap is equal to

$$\langle\!\langle[l,0,s]\otimes[l,0,s]|q^{\frac{1}{2}(L_0+\bar{L}_0)-\frac{c}{12}}|[l,0,s]\otimes[l,0,s]\rangle\!\rangle^{\sigma} = \mathrm{Tr}_{[l,0,s]\otimes[l,0,s]}\left(q^{L_0-\frac{c}{12}}\,\sigma\right). \tag{4.19}$$

Here the trace is taken in the tensor product of $\mathcal{H}_{[l,0,s]}\otimes\mathcal{H}_{[l,0,s]}$, and σ is the operator that acts on states in this tensor product by exchanging the two factors. To evaluate this trace we observe that only the diagonal terms contribute. Thus it is clear that the above overlap is proportional to $\chi_{[l,0,s]}(q^2)$. Now the subtlety concerns the fact that, depending on the value of s, we are dealing with bosonic or fermionic states. Since σ interchanges the states,

[5]This is where our analysis differs from [17].

it picks up a minus sign in the fermionic case relative to the bosonic case. Thus we find that

$$\mathrm{Tr}_{[l,0,s]\otimes[l,0,s]}\left(q^{L_0-\frac{c}{12}}\,\sigma\right) = e^{-i\pi s/2}\,\chi_{[l,0,s]}(q^2)\,. \tag{4.20}$$

With this in mind we then calculate that the overlap between a permutation and a tensor product brane equals

$$\begin{aligned}&\langle\langle L_1,S_1,L_2,S_2\| q^{\frac{1}{2}(L_0+\bar{L}_0)-\frac{c}{12}}\ \|\ [\hat{L},\hat{M},\hat{S}_1,\hat{S}_2]\rangle\rangle = \\ &\quad = \sum_{[l,m,s]}\chi_{[l,m,s]}(\tilde{q}^{1/2})\sum_{\hat{l}}\Big(N_{L_1L_2}{}^{\hat{l}}\,N_{\hat{l}\hat{L}}{}^{l}\,\delta^{(4)}(s+\hat{S}_1+\hat{S}_2-(S_1+S_2)+1)+ \\ &\qquad\qquad + N_{k-L_1L_2}{}^{\hat{l}}\,N_{\hat{l}\hat{L}}{}^{l}\,\delta^{(4)}(s+\hat{S}_1+\hat{S}_2-(S_1+S_2)-1)\Big)\,.\end{aligned} \tag{4.21}$$

We observe that the representations that appear in the open string channel are formally R-sector representations; however, they are to be interpreted as twisted NS-sector representations, where the twist is again the exchange of the two $N=2$ factors. (Note that if we had left out the factor of $e^{-i\pi s/2}$ from (4.20), then the open string representations would have had s even, and the characters could not have been interpreted in terms of twisted NS-representations; for a simple example this is explicitly demonstrated in appendix C. The fact that the twisted NS-representations are formally R-sector representations was also already realised in [38].)

The fact that s odd appears in these overlaps is also crucial from the point of view of obtaining the correct topological spectrum. The characters that appear in (4.21) are characters of a *single* $N=2$ minimal model with central charge c, evaluated at $\tilde{q}^{1/2}$. They should however be thought of as NS-characters of the diagonal $N=2$ algebra whose central charge is $2c$. Thus we should decompose them as

$$\tilde{q}^{1/2(h-c/24)}+\cdots = \tilde{q}^{h^d-c/12}+\cdots\,, \tag{4.22}$$

where h^d is the conformal dimension with respect to the diagonal algebra. Thus we find

$$h^d = \frac{1}{2}h+\frac{c}{16}\,. \tag{4.23}$$

The chiral primaries are the states for which the conformal dimension h is half the U(1)-charge q, $h=q/2$. By considering the unspecialised characters one can see that the U(1) charge of theses states (with respect to the diagonal $N=2$ algebra) is twice the U(1) charge given in (A.7); thus the chiral primaries are characterised by the property that their conformal dimension h^d must also be twice h. From (4.23) it is then easy to see that this will be precisely the case whenever $h=c/24$, i.e. whenever the representation $[l',m',s']$ is a Ramond ground state!

(Incidentally, for the case of the overlap between boundary states of opposite spin structure, i.e. for example, $\hat{S}_i=S_i+1$, the open string representations can be thought of as describing (twisted) R-sector representations. In this case s is still odd, and the chiral primaries correspond again to those representations for which $[l',m',s']$ is a Ramond ground state; indeed, these are the only states whose contribution to the open string trace is independent of $\tilde{q}$!)

Thus we have topological states whenever $m' = l' + 1$ and $s' = 1$ or $m' = -l' - 1$ and $s' = -1$. It is then clear that we get one topological boson for each representation l' that appears in the fusion product

$$l' \subset L_1 \otimes L_2 \otimes L \,. \tag{4.24}$$

Since $L \mapsto k - L$ is a simple current, the result for the fermions is the same.

This has to be compared now with the calculation of appendix B, where we only considered $L = 0$, i.e. a permutation brane with $\hat{J} = x_1 - \eta x_2$. For that case, (4.24) reduces simply to

$$|\text{bosons}| = |\text{fermions}| = \min(L_1 + 1, L_2 + 1) \,. \tag{4.25}$$

This then agrees precisely with the results of appendix B, in particular (B.20).

4.4 Flows

Finally, we can check whether the flows we found in section 3.3 from the matrix factorisation point of view are compatible with the RR-charges of the conformal field theory description. Let us first analyse the case discussed in section 3.3.1. In order to be able to compare this with the conformal field theory results we need to consider a configuration where all three rank 1 factorisations have an interpretation in terms of permutation branes (i.e. correspond to 'consecutive' lines). Translated into conformal field theory language, the rank 1 matrix flows than predict that there is a flow

$$\|[L_1, M_1, 0, 0]\rangle\!\rangle \oplus \|[L_2, M_2, 0, 0]\rangle\!\rangle \longrightarrow \|[L_1 + L_2 + 1, M_1 + L_2 + 1, 0, 0]\rangle\!\rangle \,, \tag{4.26}$$

where $M_2 - M_1 - 2 = L_1 + L_2$. However, such a flow can only exist if the RR charges of both sides agree. The Q_l charge of the left-hand side is

$$\begin{aligned} &Q_l\left(\|[L_1, M_1, 0, 0]\rangle\!\rangle\right) + Q_l\left(\|[L_2, M_2, 0, 0]\rangle\!\rangle\right) = \\ &\quad = \frac{1}{\sqrt{2}\, S_{0l}} \left(S_{L_1 l} e^{i\pi M_1 (l+1)/(k+2)} + S_{L_2 l} e^{i\pi M_2 (l+1)/(k+2)} \right) . \end{aligned} \tag{4.27}$$

Apart from the normalisation factor of the S-matrix of $su(2)_k$, the bracket is therefore

$$\begin{aligned} \sqrt{\frac{k+2}{2}}\,(\,\cdot\,) &= e^{i\pi(M_1 + L_2 + 1)(l+1)/(k+2)} \times \\ &\quad \times \left[\sin\left(\frac{\pi(L_1+1)(l+1)}{(k+2)}\right) e^{-i\pi(L_2+1)(l+1)/(k+2)} + \right. \\ &\qquad \left. + \sin\left(\frac{\pi(L_2+1)(l+1)}{(k+2)}\right) e^{i\pi(L_1+1)(l+1)/(k+2)} \right] \\ &= e^{i\pi(M_1 + L_2 + 1)(l+1)/(k+2)} \sin\left(\frac{\pi(L_1 + L_2 + 2)(l+1)}{(k+2)}\right) . \end{aligned} \tag{4.28}$$

Putting back the various factors we therefore obtain that

$$Q_l\left(\|[L_1, M_1, 0, 0]\rangle\!\rangle\right) + Q_l\left(\|[L_2, M_2, 0, 0]\rangle\!\rangle\right) = Q_l\left(\|[L_1 + L_2 + 1, M_1 + L_2 + 1, 0, 0]\rangle\!\rangle\right) . \tag{4.29}$$

Thus these matrix flows are indeed compatible with the RR charges of the conformal field theory description!

4.4.1 g-factors

We can also determine the g-factors of the various D-branes from their boundary state description. The g-factor is simply the coefficient of the Ishibashi state corresponding to $(l_i, m_i, s_i) = (0,0,0)$. For example, the g-factor of the permutation brane $\|[L, M, 0, 0]\rangle\rangle$ equals (in the following we are dropping the superfluous $(0,0)$ labels)

$$g(\|[L,M]\rangle\rangle) = \frac{1}{\sqrt{2}}\frac{S_{L0}}{S_{00}} = \frac{1}{\sqrt{2}}\frac{\sin(\pi(L+1)/(k+2))}{\sin(\pi/(k+2))}\,. \tag{4.30}$$

On the other hand, the g-factor of the tensor product brane $\|L_1, 0, L_2, 0\rangle\rangle$ is

$$\begin{aligned} g(\|L_1, L_2\rangle\rangle) &= \frac{(2k+4)}{\sqrt{2}}\frac{S_{L_100,000}\,S_{L_200,000}}{S_{000,000}} = \sqrt{k+2}\,\frac{S_{L_10}\,S_{L_20}}{S_{00}} \\ &= \sqrt{2}\,\frac{\sin(\pi(L_1+1)/(k+2))\,\sin(\pi(L_2+1)/(k+2))}{\sin(\pi/(k+2))}\,. \end{aligned} \tag{4.31}$$

In particular, we see from these expressions that the flows (4.26) that relate the permutation branes among each other are *perturbative*, i.e. that the ratio of the g-factors of the initial and final configuration approaches 1 in the limit $k \to \infty$. Indeed, the limit of the relevant ratio is in that case

$$\begin{aligned} \frac{g(\|[L_1+L_2+1, M_1+L_2+1]\rangle\rangle)}{g(\|[L_1,M_1]\rangle\rangle)+g(\|[L_2,M_2]\rangle\rangle)} &= \frac{\sin((L_1+L_2+2)\pi/(k+2))}{\sin((L_1+1)\pi/(k+2))+\sin((L_2+1)\pi/(k+2))} \\ &\overset{k\to\infty}{\longrightarrow} \frac{L_1+L_2+2}{L_1+1+L_2+1} = 1\,. \end{aligned} \tag{4.32}$$

On the other hand, the flow from a permutation brane and its anti-brane to a tensor product brane is *non-perturbative*. In fact, the g-factors of the permutation branes approach

$$g(\|[L,M]\rangle\rangle) = \frac{1}{\sqrt{2}}\frac{\sin(\pi(L+1)/(k+2))}{\sin(\pi/(k+2))} \overset{k\to\infty}{\longrightarrow} \frac{L+1}{\sqrt{2}}\,, \tag{4.33}$$

whereas

$$\lim_{k\to\infty} g(\|L_1, L_2\rangle\rangle) = 0\,. \tag{4.34}$$

Thus the ratio of the g-factors is zero in the limit $k \to \infty$. Such non-perturbative flows do not necessarily preserve the K-theory charges of the corresponding branes, and this is in fact what happens here. The permutation branes carry RR-charge and therefore integer valued K-theory charge. On the other hand, the tensor product branes carry only torsion charge (as is familiar from the case of one minimal model [39, 40]). The configuration of permutation branes that can flow to a tensor product brane does not carry any RR (or indeed K-theory) charge, and thus if the flow were to preserve the K-theory charge, it would follow that the tensor product branes would not carry any K-theory charge at all.

In fact, the situation is analogous to the case of the non-BPS D0-brane of Type I theory [41]. This D-brane can be obtained from the superposition of a D1-brane anti-D1-brane pair by taking the tachyon to be the kink solution. The resulting configuration carries non-trivial torsion $\mathbb{Z}_2$ K-theory charge. On the other hand, the original configuration of a D1-brane anti-D1-brane pair with a constant tachyon solution carries trivial K-theory charge. Changing the trivial tachyon solution to the kink therefore does not preserve the K-theory charge (since the value of the tachyon is changed at spacelike infinity).

The fact that this flow does not preserve the K-theory charge is also visible in the matrix factorisation description: the relevant tachyonic operator that needs to be switched on in order to flow from the superposition of permutation branes to the tensor product brane is not the 'constant' mode. (Indeed, as is explained in section 3.3.2, $\lambda = x_1 + \eta_n x_2$ in (3.26).) Rather, it can be thought of as being the analogue of the kink solution.

5. The Gepner Model

Now we would like to use the permutation branes analysed in the previous sections as building blocks for branes in Gepner models (see [17] for the original construction of a generalised class of such boundary states). This is the conformal field theory analog of the constructions pursued in [13, 14] from the matrix factorisation side.

The Gepner model is an orbifold of a tensor product of $N = 2$ minimal models whose central charges add up to 9. We restrict our discussion to the case of five minimal models. Together with a free field theory with $c = 3$ it describes a Calabi-Yau compactification in the light cone gauge. As opposed to the earlier sections, we work in the following with the minimal model before GSO projection. Its Hilbert space is given by

$$\mathcal{H} = \bigoplus_{[l,m,s]} \left[\left(\mathcal{H}_{[l,m,s]} \otimes \bar{\mathcal{H}}_{[l,m,s]} \right) \oplus \left(\mathcal{H}_{[l,m,s]} \otimes \bar{\mathcal{H}}_{[l,m,s+2]} \right) \right] . \tag{5.1}$$

The Hilbert space of the Gepner model before orbifolding is the tensor product of five such models, subject to the constraint that the world-sheet spin structures of the minimal models are properly aligned, i.e. that only states for which the s_i are all even, or the s_i are all odd appear. We denote the generator of the $\mathbb{Z}_{k_i+2}$ axial symmetry (4.7) in the ith minimal model by g_i. The generator of the Gepner orbifold is then $g = g_1 \cdots g_5$; its order is $H = \mathrm{lcm}\{k_i + 2\}$. In the nth twisted sector (where $n = 1, \ldots, H-1$), the right-moving $\bar{m}_i$ differ then from the left-moving m_i by $2n$ (for all $i = 1, \ldots, 5$); see [15] for explicit expressions for the partition function and [42] for a detailed discussion of the necessary projections.

We want to construct boundary states that involve the permutation boundary states for the tensor product of two minimal models. To do so, we assume that $k \equiv k_4 = k_5$. Our construction closely follows the discussion of RS boundary states [15] given in [43], to which we refer for further details including also short orbit states [42], which we will not discuss here. The original construction of permutation boundary states for more general permutation groups in Gepner models appeared previously in [17].

In each sector (NS-NS and R-R and for each spin structure η) we will consider a boundary state that is a tensor product of a standard (Neumann or Dirichlet) free field boundary state (corresponding to the free $c = 3$ theory), as well as a boundary state of the internal Gepner theory. GSO-invariance in the closed string requires that we add together states with different spin structures, while in order to obtain a GSO-invariant open string spectrum one needs to add NS-NS and R-R sector components. (For a review of these matters see for example [44].) The GSO-projection however only applies to the full ten-dimensional theory; thus the sum over the different spin-structures can only be done once all components have been tensored together.

In the following we shall only consider the internal part of each such constituent boundary state. As we have seen before, the spin structure η is related to whether the label S_i is even or odd; in the following we shall therefore always consider the case where either all S_i are even, or all S_i are odd. Furthermore, we choose the convention that NS-NS components are labelled by $s = 0$ and R-R components by $s = 1$. The constituent states for the usual and the permutation gluing conditions are described in detail in appendix D.

5.1 The transposition branes

On the basis of these expressions it is then straightforward to write down a Gepner boundary state that consists of tensor product boundary states in the first three factors, and a permutation boundary state for the last two:

$$
\begin{aligned}
&\|L_1, L_2, L_3, L, M, \hat{M}, S\rangle\rangle_{(-1)^{(s+1)F}} = \\
&= \frac{1}{\sqrt{H}} \sum_{n \in \mathbb{Z}_H} \sum_{l_1,l_2,l_3,l,m} \sum_{\nu_i \in \mathbb{Z}_2} e^{-\pi i \frac{\hat{M}n}{H}} (-1)^{S \sum \nu_i} \prod_{i=1}^{3} \left((2k_i+4)^{\frac{1}{4}} \frac{S_{L_i l_i}}{\sqrt{S_{0l_i}}} \right) \frac{1}{2} e^{\pi i \frac{Mm}{k_4+2}} \frac{S_{Ll}}{S_{0l}} \times \\
&\quad \times e^{-\pi i \frac{s}{2} S} \, |l_1, n, s+2\nu_1; l_2, n, s+2\nu_2; l_3, n, s+2\nu_3; \\
&\qquad [l, m+n, s+2\nu_4] \otimes [l, -m+n, s+2\nu_5]\rangle\rangle \, .
\end{aligned}
\tag{5.2}
$$

Here we have summed over the contributions of the twisted sectors (that are labelled by n), but not over the spin-structures, nor the NS-NS and R-R sectors — these sums can only be done once the space-time part of the boundary states has also been included.

Nevertheless, everything that is of interest can already be read off from this expression. In particular, using the formulae from appendix D (in particular (D.4) and (D.7)), the one-loop amplitude becomes

$$
\begin{aligned}
&\langle\langle L_1', L_2', L_3', L', M', \hat{M}', S' \| q^{L_0 + \bar{L}_0 - \frac{c}{12}} \| L_1, L_2, L_3, L, M, \hat{M}, S\rangle\rangle_{(-1)^{(s+1)F}} = \\
&= \frac{1}{2} \sum_{l_i, m_i, s_i} \delta^{(H)} \left(\frac{\hat{M}' - \hat{M}}{2} + H \sum_{i=1}^{5} \frac{m_i}{2k_i+4} \right) \prod_{i=1}^{5} \delta^{(2)}(S - S' + s_i) \, e^{-\frac{\pi i s}{2}(S - S' + \sum s_i)} \times \\
&\quad \times \prod_{i=1}^{3} N_{L_i' L_i}^{l_i} \sum_{l} N_{LL'}^{l} N_{l_4 l_5}^{l} \delta^{(2k+4)}(M - M' + m_4 - m_5) \prod_{i=1}^{5} \chi_{[l_i, m_i, s_i]}(\tilde{q}) \, .
\end{aligned}
\tag{5.3}
$$

Here the sums run over all quintuples of triplets (l_i, m_i, s_i) such that $l_i + m_i + s_i$ is even. (The factor of $1/2$ accounts, as in (D.7), for the fact that the equivalent representations $(l_4, m_4, s_4), (l_5, m_5, s_5)$ and $(k - l_4, m_4 + k + 2, s_4 + 2), (k - l_5, m_5 + k + 2, s_5 + 2)$ appear twice in the above expression.)

The permutation boundary states preserve one half of the space time supersymmetry, the phase of which is determined by the label $\hat{M}$, just like for the ordinary tensor product branes. Note that one can explicitly confirm from the one-loop amplitude that the spectrum on every brane is tachyon-free (once the final GSO-projection has been performed); in fact, this is simply a consequence of the fact that (5.3) depends in the usual manner on $s(S-S'+\sum_i s_i)$. Hence, the branes are stable [17].

In order to determine the charge lattice spanned by these boundary states, it is of great interest to calculate the open string Witten index $\mathrm{Tr}_R(-1)^F$ between the branes with different M and $\hat{M}$. In order to isolate this contribution from the above overlaps, one has to take the R-R component (i.e. the $s=1$ component of (5.3)), and consider the overlap between the boundary states with opposite spin structure. (Recall that in the full boundary state one has to add to the above boundary state in each sector its image under $(-1)^{F_L}$ in order to make it invariant under the closed string GSO-projection.) The action of $(-1)^{F_L}$ on each boundary state shifts each S_i by one, but also shifts $\hat{M}$ by $-H$. (This is the case both for tensor product boundary states, as well as for permutation boundary states.) Taking all this into account, one obtains

$$\begin{aligned} I(L_1', L_2', L_3', L', M', \hat{M}', S' | L_1, L_2, L_3, L, M, \hat{M}, S) = \\ = -\sum_{m_i} \delta^{(2k_4+4)}(M - M' + m_4 - m_5)\, \delta^{(H)}\left(\frac{\hat{M}' - \hat{M} + H}{2} + H \sum_{i=1}^{5} \frac{m_i}{2k_i+4}\right) \times \\ \times \prod_{i=1}^{3} \hat{N}^{m_i - 1}_{L_i' L_i} \sum_l N^l_{LL'} N^l_{m_4-1, m_5-1} e^{-\frac{\pi i}{2}(S-S')}\,, \end{aligned} \tag{5.4}$$

where $\hat{N}$ denotes the periodically continued fusion rule coefficients. Of particular interest is the basic case $L_i' = L_i = L = L' = 0$, for which the above formula simplifies further

$$\begin{aligned} I(0,0,0,0,M',\hat{M}',S'|0,0,0,0,M,\hat{M},S) = \\ = -\sum_{m_i} \delta^{(2k_4+4)}(M - M' + m_4 - m_5)\, \delta^{(H)}\left(\frac{\hat{M}' - \hat{M} + H}{2} + H \sum_{i=1}^{5} \frac{m_i}{2k_i+4}\right) \times \\ \times \left(\prod_{i=1}^{3} \hat{N}^{m_i-1}_{00}\right) N^0_{m_4-1, m_5-1}\, e^{-\frac{\pi i}{2}(S-S')}\,. \end{aligned} \tag{5.5}$$

This index is sometimes independent of the labels M, M'; in particular this is the case for $w_4 = w_5 = 1$. Along the lines of [16, 19] one can then rewrite this intersection number in terms of the symmetry generator G that shifts $\hat{M}$ by 2 (G acts as the H-dimensional shift matrix). To this end one replaces the fusion rule coefficient in each factor by $(1 - G^{-w_i})$, where w_i is the weight $w_i = H/(k_i+2)$, and accounts for the $\delta^{(H)}$-function constraint by some additional overall factor. For $w_4 = w_5 = 1$ one obtains for the index[6]

$$I_{(45)} = G^{-w_4} \prod_{i=1}^{3} (1 - G^{-w_i}) = -G^{w_4} \prod_{i=1}^{3} (1 - G^{w_i})\,. \tag{5.6}$$

It is easy to see that the same formula also holds in the case $w_4 \neq 1$, provided that $M' - M \neq 0$.

[6]For the quintic this formula was already found in [17].

In order to determine the charges of the permutation branes, it is also important to determine the overlap of a permutation brane with one of the tensor product branes. There is again a subtlety in the calculation of the overlaps between the corresponding Ishibashi states; the correct generalisation of (4.20) that is invariant under field identifications is now (we are again restricting ourselves just to two factors)

$$\begin{aligned} \langle\langle l,n,s;l,n,s|q^{L_0+\bar{L}_0-\frac{c}{24}}|[l,n,s]\otimes[l,n,s]\rangle\rangle &= \mathrm{Tr}_{[l,n,s]\otimes[l,n,s]}(\sigma\, q^{L_0-\frac{c}{24}}) \\ &= e^{\frac{\pi i n}{k+2}-\frac{\pi i s}{2}}\chi_{[l,n,s]}(q^2)\,. \end{aligned} \tag{5.7}$$

The phase factor amounts to an insertion of $(-1)^{F_L}$, which takes into account the statistics of the states that are permuted by σ.

Then it is straightforward to calculate the overlaps between tensor product and permutation branes in the Gepner model, and one finds

$$\begin{aligned} &\langle\langle L_1',\ldots,L_5',\hat{M}',S'\|q^{L_0+\bar{L}_0-\frac{c}{24}}\|L_1,L_2,L_3,L,M,\hat{M},S\rangle\rangle_{(-1)^{F(s+1)}} = \\ &= \sum_{l_i,m_i,s_i}\delta^{(H)}\left(\frac{\hat{M}'-\hat{M}+1}{2}+H\sum_{i=1}^{4}\frac{m_i}{2k_i+4}\right)\prod_{i=1}^{3}\delta^{(2)}(S-S'-s_i)\,\delta^{(2)}(s_4+1)\times \\ &\quad\times e^{-\frac{\pi i s}{2}(S-S'+\sum_{i=1}^{4}s_i+1)}\prod_{i=1}^{3}N_{L_i'L_i}^{l_i}\sum_{\hat{l}}N_{L_4'L_5'}^{\hat{l}}N_{Ll_4}^{\hat{l}}\prod_{i=1}^{3}\chi_{[l_i,m_i,s_i]}(\tilde{q})\,\chi_{[l_4,m_4,s_4]}(\tilde{q}^{\frac{1}{2}})\,. \end{aligned} \tag{5.8}$$

This overlap is in particular independent of M.

As before, the index is simply the open string Witten index $\mathrm{Tr}_R(-1)^F$. This can be calculated in the same manner as the index between two permutation branes, and one finds

$$\begin{aligned} &I(L_1',\ldots L_5',\hat{M}',S'|L_1,L_2,L_3,L,M,\hat{M},S) = \\ &\quad= -\sum_{m_i}\delta^{(H)}\left(\frac{\hat{M}'-\hat{M}+1+H}{2}+H\sum_{i=1}^{4}\frac{m_i}{2k_i+4}\right)\times \\ &\qquad\times e^{-\frac{\pi i}{2}(S-S')}\prod_{i=1}^{3}\hat{N}_{L_i'L_i}^{m_i-1}\sum_{\hat{l}}N_{L_4'L_5'}^{\hat{l}}\hat{N}_{Lm_4-1}^{\hat{l}}\,, \end{aligned} \tag{5.9}$$

where $\hat{N}$ are the periodically continued fusion rule coefficients. Since the charges of all tensor product and all permutation boundary states can be obtained by forming bound states of $L=0$ branes, we are particularly interested in the index for the brane with $L_i=L_i'=L=0$, in which case the above formula simplifies to

$$\begin{aligned} &I(0,0,0,0,0,\hat{M}',S'|0,0,0,0,M,\hat{M},S) = \\ &\quad= -\sum_{m_i}\delta^{(H)}\left(\frac{\hat{M}'-\hat{M}+1+H}{2}+H\sum_{i=1}^{4}\frac{m_i}{2k_i+4}\right)\times \\ &\qquad\times e^{-\frac{\pi i}{2}(S-S')}\left(\prod_{i=1}^{3}\hat{N}_{00}^{m_i-1}\right)\hat{N}_{0,m_4-1}^{0}\,. \end{aligned} \tag{5.10}$$

Rewriting this in terms of the symmetry operator G leads to

$$I_{(45)-RS} = -\prod_{i=1}^{4}(1-G^{-w_i}) = -G^{w_4}\prod_{i=1}^{4}(1-G^{w_i})\,. \tag{5.11}$$

5.2 The (23)(45) permutation branes

Until now we have only considered the permutation branes that involve a single transposition in the last two factors, and usual B-type boundary states for the first three factors. Using the same ingredients we can obviously also construct the boundary states where we combine two transpositions in the factors $k_2 = k_3$ and $k_4 = k_5$, with a usual B-type boundary state for the first factor. The calculations follow the same pattern as above, in particular, one just needs to collect the results of the building blocks summarised in appendix D. As before, the resulting branes are also stable. The intersection matrix between two such branes has a simple form if $w_2 = w_4 = 1$ or if both $M_1 \neq M_1'$ and $M_2 \neq M_2'$, in which case it is (for the quintic this formula was already given in [17])

$$I_{(23)(45)} = (1 - G^{w_1})\, G^{w_2}\, G^{w_4} \,. \tag{5.12}$$

For the intersection between the RS and these permutation branes we obtain on the other hand

$$I_{(23)(45)-RS} = (1 - G^{w_1})\,(1 - G^{w_2})\,(1 - G^{w_4})\, G^{w_2}\, G^{w_4} \,. \tag{5.13}$$

Finally, the intersection between the (45) and the (23)(45) boundary states is

$$I_{(23)(45)-(45)} = (1 - G^{w_1})\,(1 - G^{w_2})\, G^{w_2}\, G^{w_4} \,. \tag{5.14}$$

This last formula is again in general only correct if $w_4 = 1$ or the M-labels corresponding to the (45) permutation of the two branes are different.

6. Gepner model, matrix factorisations and geometry

At this point it is natural to ask what the large volume charges of these permutation branes are. For the RS branes, this question has first been answered in several examples in [16, 19 – 21], using the analytic continuation of periods to the Gepner point. In later work [45 – 49] it has been understood that the RS branes correspond geometrically to pull backs of certain bundles from the embedding projective space to the Calabi-Yau hypersurface.

The lattice generated by the RS branes is only a sublattice of the full charge lattice which is only in special cases of maximal rank. It also typically does not contain the charge vectors of minimal length. One may wonder whether the permutation boundary states may give rise to new charges and to charge vectors of minimal length; as we shall see in this section, both phenomena occur. In fact, at least for a number of examples (including the case of the quintic), a certain family of permutation branes can be shown to span the full charge lattice at the Gepner point. However, we also show, that there are examples where the permutation branes do not account for all charges.

6.1 The transposition branes

Using the results of [45 – 49] one can calculate the large volume charges of the RS branes in generic models. Whenever the transposition and the RS branes generate the same subspace of the RR charge lattice (this is in particular the case when $w_4 = w_5 = 1$) we can determine the charges of the transposition branes via the intersection matrix between permutation and tensor product branes. Our analysis of the flows between rank 1 factorisations and tensor product factorisations in section 3.3.2 then suggests that the RS branes can be obtained as bound states of the transposition branes, and thus that the RS charges are linear combinations (with integer coefficients) of the charges of the transposition branes. In fact, it is clear from the above equations that (at least for $w_4 = w_5 = 1$)

$$I_{(45)}\,(1 - G^{w_4}) = I_{(45)-RS}\,. \tag{6.1}$$

Thus the integer matrix $(1 - G^{w_4})$ expresses the charges of the RS branes in terms of those of the (45)-branes. This relation is also consistent with the relation between the intersection forms

$$I_{RS} = (1 - G^{w_4})^t\, I_{(45)}\,(1 - G^{w_4})\,, \tag{6.2}$$

where I_{RS} is the intersection form of the RS branes which, in our conventions, equals

$$I_{RS} = \prod_{i=1}^{5}(1 - G^{w_i})\,. \tag{6.3}$$

It also agrees with the intersection matrix involving the (23)(45) branes (if they exist),

$$I_{(23)(45)-(45)}(1 - G^{w_4}) = I_{(23)(45)-RS}\,. \tag{6.4}$$

It is remarkable that the intersection forms calculated above for the permutation branes (where the permutation is applied to two models with $w = 1$ in the Gepner model) exactly coincide with the 'intersection matrix in the Gepner basis' computed for one- and two-parameter examples in [16, 19 − 21].[7] In those papers, the Gepner basis was taken from [22 − 26], where it was obtained by analytic continuation of the fundamental period at large volume. The fundamental period is the solution of the Picard-Fuchs equations without logarithms, and corresponds to a D0 brane. Subsequently, the $\mathbb{Z}_H$ Gepner monodromy transformation was applied to this period to obtain H periods (with linear relations) at the Gepner point. By construction, the Gepner monodromy takes a particularly simple form in this basis, since it is just an H-dimensional shift matrix.

The transformation between the Gepner basis and the large volume basis was given in the old papers [22 − 26]; in particular, the fundamental period is directly mapped to one of the periods in the Gepner basis. Historically, this transformation, together with the relation between Gepner and RS basis provided the first way to determine the charges of RS branes at large volume.

Here, the relation between the RS basis and the Gepner basis is given by the same change of basis as between (45)- and RS branes. Thus we can conclude that one of the (45)-branes corresponds to the D0 brane, or the fundamental period, that has been used in the original construction of the Gepner basis. For the quintic, the relation between the (45)-boundary state and D0-branes was already mentioned in [18], where it was however claimed to correspond to 5 D0-branes.

[7] For the case of the quintic this was also noted before in [17].

CY − hypersurface	Gepner model	$I_{(12)}$	Reference
$\mathbb{P}_4[5]$	$(k=3)^5$	$-G(1-G)^3$	[22, 16, 17]
$\mathbb{P}_{(1,1,1,1,2)}[6]$	$(k=4)^4(k=1)^1$	$-G(1-G^1)^2(1-G^2)$	[24, 23, 21]
$\mathbb{P}_{(1,1,1,1,4)}[8]$	$(k=6)^4$	$-G(1-G^1)^2(1-G^4)$	[24, 23, 21]
$\mathbb{P}_{(1,1,1,2,5)}[10]$	$(k=8)^3(k=3)$	$-G(1-G)(1-G^2)(1-G^5)$	[24, 23, 21]
$\mathbb{P}_{(1,1,1,6,9)}[18]$	$(k=16)^3(k=1)$	$-G(1-G)(1-G^6)(1-G^9)$	[26, 19]
$\mathbb{P}_{(1,1,2,2,2)}[8]$	$(k=6)^2(k=2)^3$	$-G(1-G^2)^3$	[25, 20, 21]
$\mathbb{P}_{(1,1,2,2,6)}[12]$	$(k=10)^2(k=4)^2$	$-G(1-G^2)^2(1-G^6)$	[25, 20, 21]

Table 1: Intersection matrices of some Gepner models.

Table 1 gives an overview over the models where the Gepner intersection matrix was obtained by analytic continuation and shown by our analysis to coincide with the single permutation intersection matrices $I_{(45)}$, where two models with $w=1$ are permuted. (To be consistent with the labelling of the models below, these branes are more appropriately referred to as (12)-branes!) Hence, we know that D0 branes exist as rational boundary states in all of these models. To check this, we have verified by explicit counting that all of the relevant boundary states possess 3 candidate marginal operators. (In the case that the Gepner model consists of only 4 factors, one has to add into the formula for the intersection matrix a factor of $(1-G^{H/2})$ — see [56] for a discussion of the difference in the open string projection.)

6.1.1 Stability

By constructing an explicit, stable boundary state of the right charges, we have therefore also settled the question of whether the D0 brane on the quintic (and several other models) is stable at the Gepner point. This is in agreement with the analysis of the recent paper [51], where a stability condition based on matrix factorisation was formulated and applied to the case of the quintic. The existence of D0 branes in the stringy regime has been addressed in the literature from various points of view. In [16] it was noted that the position of a D0 brane would break part of the $\mathbb{Z}_5^4$ symmetry of the quintic. On the other hand, since all RS branes are invariant under this symmetry, it could not be expected to find a D0 brane among them. Rational RS boundary states carrying only (multiple units of) D0 brane charge are known to exist in other models, where this argument fails; a systematic search was performed in [52]. Since the permutation boundary states we constructed transform non-trivially under the $\mathbb{Z}_5$ symmetry scaling x_4, the symmetry is explicitly broken so that the above arguments do not apply.

In a wider context, the stability of D-branes depending on Kähler moduli has been addressed in the context of Π-stability [48, 53]. D-branes on the quintic, and in particular D0 branes, were studied in [54], who plotted the lines of marginal stability in several cases. (See also [55] for an analysis of D0 brane stability.) Their analysis shows that there exists a line of marginal stability where the D0 brane is destabilised by the D0-D6 system (that would be unstable in the large volume regime). The reason is, roughly speaking, that at the conifold point the period associated to the D6 brane shrinks to zero, so that it becomes preferable for the brane to wrap that period. The D0 brane is however stable against a decay into a D0-D6 + anti-D6 system in the stringy regime, providing a strong argument for the existence of D0 branes at the Gepner point.

6.1.2 Geometry from matrix factorisation

The interpretation of one of these boundary states as a D0-brane is also strongly suggested by the description in terms of matrix factorisations. Geometrically, D0-branes are points on the Calabi-Yau and can be described by a set of 4 linear equations $K_1 = K_2 = K_3 = K_4 = 0$, together with the defining equation of the Calabi-Yau hypersurface $W = x_1^{w_1} + x_2^{w_2} + x_3^{w_3} + x_4^{w_4} + x_5^{w_5}$.[8]

To obtain a matrix factorisation related to these equations, one can proceed as in [13] (where the case of the quintic was treated) and find four polynomials $F_1, \ldots, F_4$ such that

$$\sum_{i=1}^{4} K_i F_i = W \,. \tag{6.5}$$

Given this relation one can immediately write down a matrix factorisation, iterating the step reviewed around equation (3.4). Using the relation between matrix factorisations and the geometrical category of singularities [33] reviewed in section 3.1, one can conclude, as in [13], that such a factorisation is a good candidate for a D0 brane on the Calabi-Yau hypersurface. The location of the D0 brane is described by the zero set of the four linear equations. In particular, to make contact with our previous discussion on the relation between permutation boundary states and matrix factorisations, one can consider the special case where

$$K_1 = x_1 \,, \qquad K_2 = x_2 \,, \qquad K_3 = x_3 \,, \qquad K_4 = x_4 - \eta x_5 \,, \tag{6.6}$$

which geometrically singles out a set of points on the Calabi-Yau hypersurface. For the case of the quintic, it is shown in [13] that the deformation space of these objects is parametrised by points on the quintic, as it should be for a D0-brane.

Our detailed comparison between boundary states in the tensor product of two minimal models and matrix factorisations immediately suggests that the boundary state in the Gepner model that corresponds to this factorisation is given by the tensor product of three minimal model branes with a permutation brane in the last two factors. The labels of the boundary state should be $L_i = 0$, $L = 0$, while the choice of M corresponds to the choice of η in (6.6). This is precisely the boundary state we considered above.

The charge of the brane can be verified on the matrix side by the same type of index calculation that we performed above using conformal field theory methods. Indeed, given our detailed comparison between matrix factorisations and conformal field theory it is clear that the two calculations lead to the same result, see [13] for the quintic.

For two of the above examples, namely $\mathbb{P}_{(1,1,1,2,5)}$[10] and $\mathbb{P}_{(1,1,1,6,9)}$[18], one can easily show that the intersection matrix of the D0-brane and its images under the Gepner monodromy span already the full charge lattice. Indeed, for $\mathbb{P}_{(1,1,1,2,5)}$[10] ($\mathbb{P}_{(1,1,1,6,9)}$[19]) the

[8] Note however that in the case that the vanishing locus of the 4 linear equations intersects with an exceptional divisor, the brane will be higher dimensional; we will encounter an example of this below. If $w_4 = w_5 = 1$, this is excluded.

intersection matrix contains a 4-dimensional (6-dimensional) submatrix of determinant 1. In the other cases, however, the relevant submatrices of maximal rank have determinant bigger than 1. In these cases there exists a second construction which we will discuss in the next section.

To conclude this section, let us discuss the two-parameter example $\mathbb{P}_{(1,1,2,2,2)}[8]$ in more detail. We first summarise its geometrical properties which have been discussed in [25], to which we also refer for further details. The cohomology ring is generated by the two divisor classes L and H. L corresponds to the zero locus of degree one equations, whereas H is the zero locus of degree two equations. The two divisors intersect along a curve

$$4h = HL\,. \tag{6.7}$$

The curve $x_1 = x_2 = 0$ is a priori singular, and the singularity is resolved by blowing up each point on it into a $\mathbb{P}_1$ denoted by l,

$$4l = H^2 - 2HL\,. \tag{6.8}$$

Furthermore, we define

$$4v = \frac{H^3}{2} = H^2 L\,. \tag{6.9}$$

There are two types of (45) branes in this model: first, we can consider the equations[9]

$$x_1 = \eta x_2\,, \qquad x_3 = x_4 = x_5 = 0\,. \tag{6.10}$$

As was already mentioned before, this is geometrically a D0-brane. The second option is to consider

$$x_1 = x_2 = x_3 = 0\,, \qquad x_4 = \eta x_5\,. \tag{6.11}$$

This describes geometrically a $\mathbb{P}_1$ coming from the blow up of the singular curve. Hence, the matrix considerations suggest that this brane is a D2-brane. Indeed, calculating the brane charges via the intersections with the RS branes, one can confirm that the set of 8 (45) branes contains a single brane wrapping this cycle.

6.2 The (23)(45)-branes

The second class of branes can be constructed whenever two pairs of levels coincide. Their intersection matrix is given by (5.12) (at least if $w_2 = w_4 = 1$). Given (5.14), it is again clear that whenever such (23)(45)-branes exist, the charges of the (45)-branes can be expressed in terms of integer linear combinations (that are determined by the matrix $(1 - G^{w_2})$) of the (23)(45)-branes. Thus the latter are always more fundamental.

We will now show that the intersection matrix (5.12) always contains a submatrix of dimension $H - w_1$ which has determinant 1. First we observe that in order to determine the determinant of a submatrix (up to a sign), we can ignore the factors of $G^{w_2+w_4}$, and consider $I' = (1 - G^{w_1})$ instead. It is easy to see that this matrix has a kernel of dimension w_1; if we remove the first w_1 rows and columns the resulting matrix is upper triangular with 1s on the diagonal. Hence this submatrix of dimension $H - w_1$ has determinant 1.

[9]This brane should be more appropriately referred to as the (12)-brane.

This is already sufficient to show that the (23)(45)-branes generate the full charge lattice in the remaining examples above.[10] As before, this basis of the charge lattice is particularly natural at the Gepner point, since the Gepner monodromy just acts by a permutation. This feature is shared by the Gepner bases given in the literature (that, as we have argued, correspond to (45)-permutation boundary states); in general however, the latter are not minimally normalised.

As before, one can use the matrix description to propose a geometrical interpretation of these branes, following [14] who considered only the quintic. According to the general discussion, one is led to propose that the boundary states correspond to the zero locus of the linear equations

$$K_1 = x_1 = 0\,, \qquad K_2 = x_2 - \eta_2 x_3 = 0\,, \qquad K_3 = x_4 - \eta_4 x_5 = 0\,. \tag{6.12}$$

Generically, this system of equations describes a D2-brane. As before, one can then find F_i such that $W = \sum K_i F_i$, leading to a matrix factorisation. For the case of the quintic, the deformation theory and superpotential for these theories has been studied in [14], and known geometrical results on the obstruction theory of lines on the quintic have been reproduced.

In the case of the two-parameter model $\mathbb{P}_{(1,1,2,2,2)}[8]$, (6.12) defines the intersection of an element of the divisor class L with an element of the divisor class H. The result should therefore be a D2 brane wrapping the cycle h. This can again be verified by an explicit calculation via the index.

Since the (23)(45) branes provide a minimal basis at the Gepner point, we list their large volume charges in our two main examples. Here, one uses the charges of the RS branes, which correspond to the pure D6 brane and its monodromy images. The labelling is chosen such that the D6 has label 5. For the quintic, one obtains accordingly 5 branes whose Chern characters are

$$\begin{aligned}
\mathrm{ch}(V_1) &= 2 - H - \frac{3}{10}H^2 + \frac{7}{30}H^3 \\
\mathrm{ch}(V_2) &= -1 + H - \frac{3}{10}H^2 - \frac{7}{30}H^3 \\
\mathrm{ch}(V_3) &= \frac{1}{5}H^2 - \frac{1}{5}H^3 \\
\mathrm{ch}(V_4) &= \frac{1}{5}H^2 \\
\mathrm{ch}(V_5) &= -1 + \frac{1}{5}H^2 + \frac{1}{5}H^3\,,
\end{aligned} \tag{6.13}$$

where H is the integral generator of $H^2(M, \mathbb{Z})$ with M being the quintic. The pure D2-brane is described by V_4.

For the two parameter model $\mathbb{P}_{(1,1,2,2,2)}[8]$ we can use the list of Chern characters of the RS branes given in [45]. The cyclic ordering of the RS charges is such that the brane

[10] We have checked explicitly in these examples, that the relevant intersection matrix is indeed of the form (5.12), although $w_2 = w_4 = 1$ does not hold.

with label 8 is the pure D6-brane, while the others are its monodromy images. We find 8 branes whose Chern characters are given by

$$\begin{aligned}
\mathrm{ch}(V_1) &= -1 + h + H - l - L - \frac{5}{3}v \\
\mathrm{ch}(V_2) &= -3h + L \\
\mathrm{ch}(V_3) &= 2 - 3h - H - l + L + \frac{2}{3}v \\
\mathrm{ch}(V_4) &= h - L + v \\
\mathrm{ch}(V_5) &= -1 + h + l + v \\
\mathrm{ch}(V_6) &= h \\
\mathrm{ch}(V_7) &= h + l \\
\mathrm{ch}(V_8) &= h - v\,.
\end{aligned} \tag{6.14}$$

In this case V_6 is the pure D2-brane.

6.3 New charges from permutation branes

In general, the charge lattice spanned by the RS branes does not contain all charges of the model. In fact, the RS branes preserve a very large symmetry, and thus only relatively few Ishibashi states can be used in the construction. Geometrically, the RS branes have been identified with pullbacks of certain rigid bundles to the covering space. In general, however the cohomology of the hypersurface can be quite different from that of the embedding space; for example the torus, which has a holomorphic one-form, can be embedded as a hypersurface in $\mathbb{P}_2$, which does not have a one-form.

In this section, we will study the example $\mathbb{P}_{(1,1,1,3,3)}[9]$, or, as a Gepner model, $(k = 7)^3(k = 1)^2$. In this example, the permutation branes give rise to new charges, as one can see from a calculation of the rank of the intersection matrices. The model has $h^{(1,1)} = 4$, such that we expect four D2 and four D4 branes. Together with the D0 and D6 brane, they span a 10 dimensional charge lattice. It was realised in [57] that the RS branes do not carry all of these charges, but that one can project out the missing ones by taking a suitable free orbifold, leading to a two-parameter model with torsion in K-theory.

We can use the formulae of chapter 5 to verify that the intersection matrix of the (45)-branes, which in this case equals (for $M = M'$)

$$I_{(45)} = (1 + G^6)(1 - G)^3 \tag{6.15}$$

has rank 8, while that of the RS branes only has rank 6. Note that in this case $w_4 = 3$, so that the above intersection form is not of the standard form (5.6) but was derived directly from the conformal field theory formula (5.5). We have also calculated the intersection matrix for the (12)(45) branes which turns out to be

$$I_{(12)(45)} = -G(1 + G^6)(1 - G)\,, \tag{6.16}$$

and also has rank 8. In order to account for the full charge lattice one observes that the intersection matrix of the (45) branes at $M = 0$ and $M = 2$ has actually rank 10. [This intersection matrix is of the form

$$I_{(45);0,2} = \begin{pmatrix} (1+G^6)(1-G)^3 & -G^3(1-G)^3 \\ G^{-3}(1-G^{-1})^3 & (1+G^6)(1-G)^3 \end{pmatrix}, \tag{6.17}$$

as follows from (6.15) as well as from (5.6).] The same is the case for the intersection matrix of the (12)(45) branes at $M_1 = M_2 = 0$ and $M_1 = M_2 = 2$; in the latter case, this intersection matrix contains a submatrix of determinant 1. [The (45) branes only generate a sublattice of index 9.] Thus the full charge lattice is generated by the (12)(45) branes at $M_1 = M_2 = 0$ and $M_1 = M_2 = 2$ (and all values of $\hat{M}$).

The fact that the permutation branes generate extra charges fits nicely with expectations from geometry and matrix factorisations as we will now explain. Due to the divisibility properties of the weights, the embedding projective space has a $\mathbb{Z}_3$ singularity that locally looks like $\mathbb{C}^3/\mathbb{Z}_3$. This singularity is resolved by an exceptional $\mathbb{P}_2$. The hypersurface intersects with the singular locus in the three points $x_4^3 + x_5^3 = 0$. These 3 points are cut out and replaced by exceptional divisors, leading altogether to $h^{(1,1)} = 4$ for the cohomology of the hypersurface [57]. We can single out the singular points by the equations

$$x_1 = x_2 = x_3 = 0\,, \qquad x_4 - \eta x_5 = 0\,. \tag{6.18}$$

One would then expect that the corresponding matrix factorisation, constructed as in section 6.1 and 6.2, give a matrix description of branes wrapping the exceptional divisors, or equivalently, that the three (45) transposition branes with different M labels carry the relevant charges. This is in agreement with the calculation of the rank of the intersection matrix mentioned above. On the other hand, for fixed M the rank of the intersection matrix is 8, thus accounting for the 2 D4 charges (+ 2 D2 +D0+D6) already carried by the RS branes, as well as one extra D4 (+ D2) brane charge.

As a further confirmation of this picture one can match the symmetry transformations of the branes. Consider the action

$$(x_1, x_2, x_3, x_4, x_5) \to (x_1, x_2, x_3, e^{\frac{2\pi i}{3}} x_4, e^{\frac{4\pi i}{3}} x_5)\,. \tag{6.19}$$

This transforms the equation (6.18) with different values for η into each other, so that only their superposition remains invariant. On the level of boundary states, one sees that the RS states remain invariant under the corresponding action in conformal field theory, whereas the label M of the permutation branes gets shifted as $M \to M + 2$. This means that only the invariant combination can carry a charge contained in the RS lattice.

One may wonder whether permutation branes whose permutations involve longer cycles could generate additional charges or describe other preferred bases. This does not seem to be the case. In fact, one can easily see that the number of B-type Ishibashi states corresponding to permutations with longer cycles is always smaller than that of a permutation whose longest cycles have length two. In this sense the above constructions are already ‘optimal’, and at least as far as charges are concerned, it is sufficient to consider only transpositions as we have done in this paper.

Finally one may wonder whether the permutation branes always generate the full charge lattice. It is however easy to see that this is not the case. A simple example is the manifold $\mathbb{P}_{(3,3,4,6,8)}$[24], or, as a Gepner model, $(k=6)^2(k=4)(k=2)(k=1)$. This theory has only one class of permutation branes (the (12)-branes), and one can easily show that they generate a sublattice of rank 12. On the other hand, $h^{1,1}=7$ [58, 59], and thus the full charge lattice has dimension 16. In fact, one can identify certain RR ground states of this theory which are part of the even cohomology charge lattice, but which cannot couple to *any* standard tensor product or permutation branes.[11] For this theory, these symmetric constructions therefore do not account for all the charges. The situation is therefore similar to the case of WZW models of groups of higher rank (see for example [60, 61]).

From the matrix factorisation point of view, one can guess how to obtain the remaining constructions — these correspond probably to factorisations that rely on the fact that other pairs of w_i have common factors. This is also what one expects from geometry, since common factors of the weights lead to additional exceptional sets. At this stage it is however not clear what the corresponding boundary states in conformal field theory should be.

7. Conclusion

In this paper we have identified the D0-brane and D2-brane for a number of Gepner models with certain permutation boundary states [17]. In some examples these D-branes, together with their images under the Gepner monodromy, form a basis for the full charge lattice. We have also shown that the permutation branes sometimes carry charges that are not already accounted for by the RS branes. In general, however, the permutation branes are not yet sufficient to describe all the charges.

Our analysis was inspired by the identification of the D0- and D2-brane for the quintic in terms of matrix factorisations that was given in [13, 14]. In particular, we studied the dictionary between factorisations of $W = x_1^d + x_2^d$ with boundary states of the tensor product of two minimal models. This allowed us to identify the relevant factorisations with permutation boundary states, and thus to identify the boundary states of the D0- and the D2-brane. We also checked that these boundary states have the correct charges by determining the Witten index with the RS boundary states whose charges had been known before. (For the case of the quintic, this had also been done, from the matrix point of view, in [13, 14].) Our analysis shows in particular, that the D0-brane (as well as the D2-brane) is stable at the Gepner point.

In the examples we considered it was possible to predict the charge of one out of the H monodromy images from the form of the factorisation, which indicated the location of the brane as the zero set of linear equations. It would be interesting to generalise this, including the effect of the GSO projection (choice of grading in the matrix factorisation). This could presumably be done in the framework of the linear sigma model where one can interpolate between large and small volume, generalising the analysis for fractional branes in [45–47, 49].

[11] We thank Stefan Fredenhagen for helping us find this example and check this property.

As is explained in detail in section 4, we were only able to identify a subset of rank 1 factorisations with permutation boundary states; it would clearly be interesting to understand the boundary state description of the remaining factorisations. For the theories where the permutation branes do not account for the full charge lattice, it would also be interesting to understand (both from the matrix factorisation point of view as well as in conformal field theory) the branes that generate the remaining charges.

Concerning the relation between conformal field theory and matrix factorisations, it would be interesting to understand conceptually the relation between the various factorisations and their boundary states; in particular, one may hope to be able to read off the symmetries that are preserved by the brane (i.e. in particular the gluing condition) from the structure of the factorisation. Among other things, this should clarify how the factorisations that correspond to the permutation branes are singled out, and lead to clues for how to find the boundary state description of the remaining factorisations. A further example, where a better understanding of symmetries might be useful, are higher order permutation branes and their relation to matrix factorisations.

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A. Conventions

In this appendix we collect our conventions for the description of the $N = 2$ minimal models. The $N = 2$ algebra is generated by the modes L_n, J_n, $G^{\pm}_r$, subject to the commutation relations

$$\begin{aligned}
[L_m, L_n] &= (m-n)\, L_{m+n} + \frac{c}{12}\, m\,(m^2-1)\,\delta_{m,-n} \\
[L_m, J_n] &= -n\, J_{m+n} \\
[L_m, G^{\pm}_r] &= \left(\frac{m}{2} - r\right) G^{\pm}_{m+r} \\
[J_m, G^{\pm}_r] &= \pm\, G^{\pm}_{m+r} \\
[J_m, J_n] &= \frac{c}{3}\, m\,\delta_{m,-n} \\
\{G^+_r, G^-_s\} &= 2\, L_{r+s} + (r-s)\, J_{r+s} + \frac{c}{3}\left(r^2 - \frac{1}{4}\right)\delta_{r,-s}\,.
\end{aligned}$$

Here m and n are integer; r is integer in the R-sector, and of the form $r = \mathbb{Z} + \frac{1}{2}$ in the NS-sector. The $N = 2$ minimal models occur for

$$c = \frac{3k}{k+2}\,, \tag{A.1}$$

where k is a positive integer. At least in this case, the bosonic subalgebra of the $N=2$ algebra can be described in terms of the coset

$$(N=2)_{\text{ bos}} = \frac{su(2)_k \oplus u(1)_4}{u(1)_{2k+4}}\,. \tag{A.2}$$

Here $u(1)_d$ describes the U(1) theory whose representations are labelled by integers mod d. The central charge of (A.2) obviously agrees with (A.1).

The representations of the coset algebra are labelled by (l,m,s), where $l=2j$ with j the (half-integer valued) spin of $su(2)$, $m\in\mathbb{Z}_{2k+4}$ and $s\in\mathbb{Z}_4$. Since the $su(2)$ affine algebra appears at level k, l takes the values $l=0,1,\ldots,k$. These labels are subject to the selection rule $l+m+s=0$ mod 2. Furthermore we have the field identification

$$(l,m,s)\sim(k-l,m+k+2,s+2)\,. \tag{A.3}$$

The corresponding equivalence class will be denoted by $[l,m,s]$. We shall usually suppress the level k in our notation; also the central charge c will always be defined by (A.1).

Since the coset algebra only describes the bosonic subalgebra of the $N=2$ algebra, the irreducible representations of the $N=2$ algebra consist of direct sums of representations of the coset algebra. In fact, the NS-representations of the $N=2$ algebra correspond to the sums

$$(l,m)=(l,m,0)\oplus(l,m,2)\,, \tag{A.4}$$

where $l+m$ is even, and $(l,m)\sim(k-l,m+k+2)$. The R-representations of the $N=2$ algebra correspond on the other hand to

$$(l,m)=(l,m,1)\oplus(l,m,3)\,, \tag{A.5}$$

where $l+m$ is odd, and $(l,m)\sim(k-l,m+k+2)$. In either case, we also denote the corresponding equivalence class by $[l,m]$.

The conformal weights and the U(1)-charge (of the $N=2$ U(1) generator) of the highest weight states of the coset representation (l,m,s) are, up to integers, given by

$$h(l,m,s) = \frac{l(l+2)-m^2}{4(k+2)}+\frac{s^2}{8} \tag{A.6}$$

$$q(l,m,s) = \frac{s}{2}-\frac{m}{k+2}\,. \tag{A.7}$$

In the NS sector, the chiral primary states appear in the representations $(l,l,0)$ or $(l,-l-2,2)$. (Note that $(l,l,0)\sim(l,-l-2,2)$.) In the R sector, the condition for a chiral primary state is $(l,l+1,1)$ or $(l,-l-1,-1)$. Finally, the modular S-matrix of the coset theory is

$$S_{LMS,lms}=S_{Ll}\,\frac{1}{\sqrt{2k+4}}\,e^{i\pi\frac{Mm}{k+2}}\,e^{-i\pi\frac{Ss}{2}}\,. \tag{A.8}$$

Here S_{Ll} denotes the S-matrix of $su(2)_k$, which is explicitly given as

$$S_{Ll}=\sqrt{\frac{2}{k+2}}\,\sin\left(\pi\,\frac{(L+1)\,(l+1)}{k+2}\right). \tag{A.9}$$

One easily checks that the S-matrix (A.8) is unitary, and that the above definition depends only on the equivalence class $[L,M,S]$ and $[l,m,s]$. One also easily observes that

$$S_{LMS,lms+2}=(-1)^S\,S_{LMS,lms}\,. \tag{A.10}$$

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B. Open string spectra from matrix factorisations

In this appendix we give details of the calculations to determine the topological open string spectra between various matrix factorisations of the product theory.

B.1 The open string spectrum between rank 1 branes

In the main part of the paper we only discussed the case where the two branes in question are the same. Here we describe the calculation in the general case, where

$$J = \prod_{m\in I} (x_1 - \eta_m x_2)\,, \qquad E = \prod_{n\in D\setminus I} (x_1 - \eta_n x_2)\,, \tag{B.1}$$

and

$$\hat{J} = \prod_{\hat{m}\in \hat{I}} (x_1 - \eta_{\hat{m}} x_2)\,, \qquad \hat{E} = \prod_{\hat{n}\in D\setminus \hat{I}} (x_1 - \eta_{\hat{n}} x_2)\,. \tag{B.2}$$

We start with the discussion of the fermionic spectrum. Dividing the first line of the BRST-invariance condition

$$\begin{aligned} 0 &= \hat{E}\, t_0 + t_1\, J \\ 0 &= \hat{J}\, t_1 + t_0\, E \end{aligned} \tag{B.3}$$

by the greatest common divisor of $\hat{E}$ and J, which is $\prod_{n\in I\setminus\{I\cap\hat{I}\}}(x_1 - \eta_n x_2)$, we obtain

$$t_0 = b(x_1,x_2) \prod_{n\in I\cap\hat{I}} (x_1 - \eta_n x_2)\,, \qquad t_1 = -b(x_1,x_2) \prod_{n\in D\setminus\{I\cup\hat{I}\}} (x_1 - \eta_n x_2)\,. \tag{B.4}$$

At this point, $b(x_1,x_2)$ is an arbitrary polynomial that will get constrained by demanding that this solution is not BRST exact. BRST exact solutions are of the form $t_0 = \hat{J}\phi_0 - \phi_1 J$, so that

$$b(x_1,x_2) \in \frac{\mathbb{C}[x_1,x_2]}{\langle s_1, s_2\rangle}\,, \tag{B.5}$$

where

$$s_1 = \prod_{n\in I\setminus\{I\cap\hat{I}\}} (x_1 - \eta_n x_2)\,, \qquad s_2 = \prod_{n\in \hat{I}\setminus\{I\cap\hat{I}\}} (x_1 - \eta_n x_2)\,. \tag{B.6}$$

Since s_1 and s_2 do not have common factors, the dimension of this ring is

$$\deg s_1 \cdot \deg s_2 = |I \setminus \{I\cap\hat{I}\}| \cdot |\hat{I} \setminus \{I\cap\hat{I}\}|\,. \tag{B.7}$$

This concludes the counting of the fermions. Note that this is just the number of intersections of the set of lines

$$\bigcup_{n\in I\setminus\{I\cap\hat{I}\}} \{x_1 - \eta_n x_2\}$$

with the set of lines

$$\bigcup_{n\in \hat{I}\setminus\{I\cap\hat{I}\}} \{x_1 - \eta_n x_2\}\,,$$

which is what the geometrical picture predicts.

Turning to the bosonic spectrum, we divide the first BRST invariance condition

$$\begin{aligned} 0 &= \hat{J}\,\phi_0 - \phi_1\,J \\ 0 &= \hat{E}\,\phi_1 - \phi_0\,E \end{aligned} \tag{B.8}$$

by the greatest common divisor of J and $\hat{J}$. Then we find that

$$\phi_1 = a(x_1,x_2) \prod_{n\in\hat{I}\backslash\{I\cap\hat{I}\}} (x_1-\eta_n x_2)\,, \qquad \phi_0 = a(x_1,x_2) \prod_{n\in I\backslash\{I\cap\hat{I}\}} (x_1-\eta_n x_2)\,. \tag{B.9}$$

For this not to be BRST exact, the polynomial a has to be in the quotient ring of $\mathbb{C}[x_1,x_2]$ by the ideal generated by s_1' and s_2', where

$$s_1' = \prod_{n\in I\cap\hat{I}} (x_1-\eta_n x_2)\,, \qquad s_2' = \prod_{n\in D\backslash\{I\cup\hat{I}\}} (x_1-\eta_n x_2)\,. \tag{B.10}$$

This means that the number of bosons equals

$$|\text{bosons}| = \deg s_1' \deg s_2' = |I\cap\hat{I}||D\setminus\{I\cup\hat{I}\}|\,. \tag{B.11}$$

B.2 The spectrum between a tensor product and a rank 1 brane

To compute the spectrum between the tensor product brane Q and the rank 1 brane $\hat{Q}$, we have to do again essentially the same computation as before for the case of a single minimal model, except that now the morphisms (ϕ_0,ϕ_1) are two-component row vectors, $\phi_0 = (\phi_0^1,\phi_0^2)$ and $\phi_1 = (\phi_1^1,\phi_1^2)$. Likewise the fermions (t_0,t_1) have now also two components, $t_0 = (t_0^1,t_0^2)$ and $t_1 = (t_1^1,t_1^2)$. We will restrict ourselves to the case of a permutation brane consisting of a single line ($\hat{J}$ linear), and an arbitrary tensor product brane labelled by $(\ell_1 = L_1+1, \ell_2 = L_2+1)$. (The case $\ell_1=\ell_2=1$ was treated in [13].) The BRST conditions for the bosons are

$$\phi_1^1 x_2^{\ell_2} + \phi_1^2 x_1^{d-\ell_1} - \hat{J}\phi_0^1 = 0 \tag{B.12}$$
$$\phi_1^1 x_1^{\ell_1} - \phi_1^2 x_2^{d-\ell_2} - \hat{J}\phi_0^2 = 0 \tag{B.13}$$
$$\phi_0^1 x_2^{d-\ell_2} + \phi_0^2 x_1^{d-\ell_1} - \hat{E}\phi_1^1 = 0 \tag{B.14}$$
$$\phi_0^1 x_1^{\ell_1} - \phi_0^2 x_2^{\ell_2} - \hat{E}\phi_1^2 = 0\,. \tag{B.15}$$

We can formally solve the first two equations by

$$\begin{aligned} \phi_0^1 &= \frac{1}{\hat{J}}\left(\phi_1^1 x_2^{\ell_2} + \phi_1^2 x_1^{d-\ell_1}\right) \\ \phi_0^2 &= \frac{1}{\hat{J}}\left(\phi_1^1 x_1^{\ell_1} - \phi_1^2 x_2^{d-\ell_2}\right). \end{aligned} \tag{B.16}$$

It is easy to see that this solution also solves the third and fourth equation. We now have to discuss under which conditions the expressions (B.16) are polynomials. This will be the case whenever the zeros of the polynomial in the numerator contain those of the denominator, which translates to the following relation between ϕ_1^1 and ϕ_1^2

$$\phi_1^1 = -\phi_1^2 \eta^{d-\ell_1-k} x_1^k x_2^{d-\ell_1-\ell_2-k} , \tag{B.17}$$

where η is the root that appears in $\hat{J} = x_1 - \eta x_2$, and k is an arbitrary integer. In particular, we therefore see that for a given ϕ_1^2 there are a number of choices for ϕ_1^1. Once ϕ_1^1 and ϕ_1^2 are chosen, the remaining components ϕ_0^1 and ϕ_0^2 are determined uniquely by (B.16). These bosonic degrees of freedom are BRST trivial if

$$\phi_1^1 = t_0^1 x_2^{d-\ell_2} + t_0^2 x_1^{d-\ell_1} + (x_1 - \eta x_2) t_1^1 \tag{B.18}$$

$$\phi_1^2 = t_0^1 x_1^{\ell_1} - t_0^2 x_2^{\ell_2} + (x_1 - \eta x_2) t_1^2 . \tag{B.19}$$

It is convenient to introduce the new coordinates $z = x_1 - \eta x_2$ and x_2. Choosing different polynomials t_0^1, t_0^2, we can eliminate the z-dependence of ϕ_1^1 and ϕ_1^2. As a consequence, ϕ_1^1 and ϕ_1^2 are effectively polynomials of one variable only. In particular, the choices of ϕ_1^1 for a given ϕ_1^2 in (B.17) are all equivalent, such that all components $\phi_0^1, \phi_0^2, \phi_1^1$ are uniquely determined by ϕ_1^2. The possible choices for ϕ_1^2 are constrained by (B.19), from which it follows that

$$\phi_1^2 \in \mathbb{C}[x_2]/\langle x_2^{\ell_{min}} \rangle , \quad \text{where} \quad \ell_{min} = \min\{\ell_1, \ell_2\} . \tag{B.20}$$

The fermionic spectrum can be determined similarly. The BRST conditions are

$$t_1^1 x_2^{\ell_2} + t_1^2 x_1^{d-\ell_1} + \hat{E} t_0^1 = 0 \tag{B.21}$$

$$t_1^1 x_1^{\ell_1} - t_1^2 x_2^{d-\ell_2} + \hat{E} t_0^2 = 0 \tag{B.22}$$

$$t_0^1 x_2^{d-\ell_2} + t_0^2 x_1^{d-\ell_1} + \hat{J} t_1^1 = 0 \tag{B.23}$$

$$t_0^1 x_1^{\ell_1} - t_0^2 x_2^{\ell_2} + \hat{J} t_1^2 = 0 . \tag{B.24}$$

In analogy to the discussion of the bosons, one solves the last two equations for t_1^1 and t_1^2 and shows that the first two equations are then automatically satisfied. From the requirement that the formal solutions are in fact polynomials we derive

$$t_0^1 = -\eta^{d-\ell_1-k} x_2^{\ell_2-\ell_1-k} x_1^k t_0^2 , \tag{B.25}$$

where we have made the additional assumption that $\ell_1 \leq \ell_2$, and used that $\eta^d = -1$. The operator is BRST exact if

$$t_0^1 = -\phi_1^1 x_2^{\ell_2} - \phi_1^2 x_1^{d-\ell_1} + (x_1 - \eta x_2) \phi_0^1 \tag{B.26}$$

$$t_0^2 = -\phi_1^1 x_1^{\ell_1} + \phi_1^2 x_2^{d-\ell_2} + (x_1 - \eta x_2) \phi_0^2 . \tag{B.27}$$

Choosing coordinates z, x_2 as above, it can be seen that the z-dependence of t_0^1 and t_0^2 can be eliminated. The highest power of x_2 for t_0^2 is $\ell_1 - 1$; since we have assumed that $\ell_1 \leq \ell_2$ this equals $\min\{\ell_1, \ell_2\} - 1$. Hence, the number of fermions propagating between the two branes is $\min\{\ell_1, \ell_2\}$ and equals the number of bosons. This was to be expected since the tensor product branes are uncharged and therefore the Witten index with any other brane is zero.

C. Twisted NS-representations

As a simple example we consider the case of the tensor product of two $N = 2$ minimal models with $k = 1$. For $k = 1$ (for which $c = 1$) the conformal weights of the NS representations are $h = 0$ (for $[0, 0, 0]$) and $h = 1/6$ (for $[1, \pm 1, 0]$). If the characters that appeared in the overlap between the permutation and the tensor product brane were NS-characters evaluated at $\tilde{q}^{1/2}$, then we would have

$$\chi_{[0,0,0]}(\tilde{q}^{1/2}) = \tilde{q}^{-1/48}\left(1 + \mathcal{O}(\tilde{q}^{1/2})\right) = \tilde{q}^{h^d - 1/12}\left(1 + \mathcal{O}(\tilde{q}^{1/2})\right), \qquad h^d = \frac{1}{16} \tag{C.1}$$

$$\chi_{[1,\pm1,0]}(\tilde{q}^{1/2}) = \tilde{q}^{1/16}\left(1 + \mathcal{O}(\tilde{q}^{1/2})\right) = \tilde{q}^{h^d - 1/12}\left(1 + \mathcal{O}(\tilde{q}^{1/2})\right), \qquad h^d = \frac{7}{48}. \tag{C.2}$$

These characters now have to be interpreted as NS-characters of the diagonal $N = 2$ algebra, whose central charge is $c = 2$. This is again a minimal model (with $k = 4$), and therefore the allowed conformal weights are known. One easily checks that neither $h^d = 1/16$ nor $h^d = 7/48$ are among the allowed list of conformal weights. Thus the above characters *cannot* be interpreted in terms of NS-representations of the diagonal $N = 2$ algebra.

The situation is different if the characters that appear in the overlap between the permutation and the tensor product brane are R-characters (as we have argued they must). The conformal weights of the R-representations for $k = 1$ are $h = 1/24$ (for $[0, \pm 1, 1]$) and $h = 3/8$ (for $[1, 0, 1]$). Instead of (C.1) and (C.2) we then have

$$\chi_{[0,\pm1,1]}(\tilde{q}^{1/2}) = \tilde{q}^{0}\left(1 + \mathcal{O}(\tilde{q}^{1/2})\right) = \tilde{q}^{h^d - 1/12}\left(1 + \mathcal{O}(\tilde{q}^{1/2})\right) \qquad h^d = \frac{1}{12}, \tag{C.3}$$

$$\chi_{[1,0,1]}(\tilde{q}^{1/2}) = \tilde{q}^{1/6}\left(1 + \mathcal{O}(\tilde{q}^{1/2})\right) = \tilde{q}^{h^d - 1/12}\left(1 + \mathcal{O}(\tilde{q}^{1/2})\right) \qquad h^d = \frac{1}{4}. \tag{C.4}$$

The corresponding diagonal weights are then indeed allowed conformal weights for the $k = 4$ theory: $h = 1/12$ is the conformal weight of the NS-representation $[1, \pm 1, 0]$, while $h = 1/4$ is the conformal weight of the NS-representation $[3, \pm 3, 0]$.

D. Gepner construction

As is explained in the main part of the paper, the boundary states for the Gepner model can be made by tensoring together constituent boundary states of the branes in the $N = 2$ minimal model before GSO projection, followed by an orbifold projection. For a single minimal model, the B-type boundary state is

$$\|L, S\rangle\rangle_{(-1)^{(s+1)F}} = (2k+4)^{1/4} e^{-\pi i \frac{Ss}{2}} \sum_{l} \sum_{\nu \in \mathbb{Z}_2} \frac{S_{Ll}}{\sqrt{S_{0l}}} (-1)^{S\nu} |[l, 0, s + 2\nu]\rangle\rangle \tag{D.1}$$

where $s = 0$ for the NS-NS and $s = 1$ for the R-R sector, and the sum runs over all l such that $l + s$ is even. In the above notation, the subscript refers to the periodicity conditions of the closed string fields on the circle; thus $s = 0$ corresponds to the NS-NS sector, while $s = 1$

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is the R-R sector. Note that we can recover the boundary states constructed in section 2 by adding a NS-NS boundary state of the unprojected theory and a R-R boundary state with the appropriate normalisation factor of $1/\sqrt{2}$.

The boundary state is invariant under the $\mathbb{Z}_{k+2}$ axial symmetry of the minimal model. For the Gepner construction we need to formulate the boundary states also in the sector twisted by g^n, where g is the generator of the axial symmetry

$$\|L, \hat{M}, S\rangle\rangle_{(-1)^{(s+1)F} g^n} = (2k+4)^{\frac{1}{4}} e^{-\pi i \frac{\hat{M}n}{k+2} - \pi i \frac{Ss}{2}} \sum_{l} \sum_{\nu \in \mathbb{Z}_2} \frac{S_{Ll}}{\sqrt{S_{0l}}} (-1)^{S\nu} |l, n, s+2\nu\rangle\rangle \,. \quad \text{(D.2)}$$

Here $|l, n, s+2\nu\rangle\rangle$ is the Ishibashi state in the sector $\mathcal{H}_{[l,n,s+2\nu]} \otimes \bar{\mathcal{H}}_{[l,-n,-s-2\nu]}$, and the sum runs only over those values of l for which $l+n+s$ is even. In the open string sector, the twist leads to an insertion of g^n. It is then clear that we need a further label that specifies the action of the global symmetry on the Chan-Paton labels. This is the origin of the label $\hat{M}$ that appears only in the overall phase of the twisted boundary state. In order for this formula to make sense, we need that $L + \hat{M} + S$ is even.

The one-loop overlap between two such states is then (in the sector labelled by s and n)

$$\begin{aligned} &\langle\langle L', \hat{M}', S' \| q^{\frac{1}{2}(L_0 + \bar{L}_0) - \frac{c}{24}} \| L, \hat{M}, S\rangle\rangle_{(-1)^{(s+1)F} g^n} = \\ &= \sum_{[l', m', s']} \delta^{(2)}(S + S' + s')\, e^{i\pi \frac{n}{k+2}(\hat{M}' - \hat{M} + m')}\, e^{i\pi \frac{s}{2}(S' - S - s')} \times \\ &\quad \times \left(N_{L'L}^{l'} + (-1)^{n+s} N_{L'L}^{k-l'} \right) \chi_{[l', m', s']}(\tilde{q}) \,. \end{aligned} \quad \text{(D.3)}$$

Here the sum runs over all equivalence classes $[l', m', s']$. If we sum over all triples (l', m', s') such that $l' + m' + s'$ is even, we can instead write the last expression as

$$\begin{aligned} &\langle\langle L', \hat{M}', S' \| q^{\frac{1}{2}(L_0 + \bar{L}_0) - \frac{c}{24}} \| L, \hat{M}, S\rangle\rangle_{(-1)^{(s+1)F} g^n} = \\ &= \sum_{(l', m', s')} \delta^{(2)}(S + S' + s')\, e^{i\pi \frac{n}{k+2}(\hat{M}' - \hat{M} + m')}\, e^{i\pi \frac{s}{2}(S' - S - s')}\, N_{L'L}^{l'}\, \chi_{[l', m', s']}(\tilde{q}) \,. \end{aligned} \quad \text{(D.4)}$$

The tensor product boundary states of [15] can in this notation be written as

$$\begin{aligned} &\|L_1, \ldots, L_5, \hat{M}, S\rangle\rangle_{(-1)^{(s+1)F}} \\ &= \frac{1}{\sqrt{H}} \sum_{n \in \mathbb{Z}_H} \bigotimes_{i=1}^{5} \|L_i, \hat{M}_i, S_i\rangle\rangle_{(-1)^{(s+1)F} g^n} \\ &= \frac{\prod_i (2k_i + 4)^{\frac{1}{4}}}{\sqrt{H}} \sum_{n \in \mathbb{Z}_H} e^{-\pi i \sum_i \left(\frac{\hat{M}_i n}{k_i + 2} + \frac{S_i s}{2} \right)} \bigotimes_{i=1}^{5} \sum_{l_i} \sum_{\nu_i \in \mathbb{Z}_2} \frac{S_{L_i l_i}}{\sqrt{S_{0 l_i}}} (-1)^{S_i \nu_i} |l_i, n, s + 2\nu_i\rangle\rangle \,. \end{aligned}$$

Note that the boundary state on the left actually only depends on $\hat{M} = H \sum_i \frac{\hat{M}_i}{k_i+2}$. Furthermore, since the S_i are either all even or all odd (so that we choose the same sign for the gluing condition η in all five factors), the boundary state actually only depends on $S = \sum_i S_i$.

For the permutation boundary states the analysis is similar. Before GSO-projection the permutation boundary state in the tensor product of two minimal models is given by

$$\|L,M,S_1,S_2\rangle\!\rangle_{(-1)^{(s+1)F}} = \frac{1}{2}\sum_{l,m}\sum_{\nu_1,\nu_2\in\mathbb{Z}_2}\frac{S_{Ll}}{S_{0l}}e^{\pi i\frac{Mm}{k+2}}\,(-1)^{S_1\nu_1+S_2\nu_2}\,e^{-\pi i\frac{s}{2}(S_1+S_2)}\times$$
$$\times\,|[l,m,s+2\nu_1]\otimes[l,-m,s+2\nu_2]\rangle\!\rangle\,. \qquad \text{(D.5)}$$

Here $L+M$ and S_1+S_2 are even, and the sum runs over all l,m such that $l+m+s$ is even. As before this boundary state is invariant under $g_1\,g_2$, but not under g_1 or g_2 individually, which shift the M-label by ± 2. To construct the permutation boundary states in the sectors twisted by $g = g_1g_2$, we observe that the permutation gluing condition requires that $\bar{m}_2 = -m_1$ and $m_2 = -\bar{m}_1$. In the sector twisted by g^n the relation between left and right-moving m-labels is $m_1 = \bar{m}_1 + 2n$, $m_2 = \bar{m}_2 + 2n$, so that the relevant Ishibashi states have labels $m_2 = -\bar{m}_1 = -m_1 + 2n$. Therefore, the boundary state takes the form

$$\|L,M,\hat{M},S_1,S_2\rangle\!\rangle_{(-1)^{(s+1)F}g^n} = \frac{1}{2}e^{-\frac{\pi in}{k+2}(M+\hat{M})}\sum_{l,m}\sum_{\nu,\nu_2\in\mathbb{Z}_2}\frac{S_{Ll}}{S_{0l}}\,e^{\pi i\frac{Mm}{k+2}}\,(-1)^{S_1\nu_1+S_2\nu_2}\times$$
$$\times e^{-\pi i\frac{s}{2}(S_1+S_2)}\,|[l,m,s+2\nu_1]\otimes[l,-m+2n,s+2\nu_2]\rangle\!\rangle\,,$$

where, as before, an additional label $\hat{M}$ had to be introduced. We require that $M+\hat{M}$ is always even, so that the boundary state is invariant under $n\to n+k+2$. Also, as before in the discussion of section 4, $L+M$ and S_1+S_2 are even.

With this ansatz we then obtain the following one-loop amplitudes between twisted permutation boundary states (in the sector labelled by s and n)

$$\langle\!\langle L',M',\hat{M}',S_1',S_2'\|q^{\frac{1}{2}(L_0+\bar{L}_0)-\frac{c}{12}}\|L,M,\hat{M},S_1,S_2\rangle\!\rangle_{(-1)^{(s+1)F}g^n} =$$
$$= \sum_{[l_1,m_1,s_1],[l_2,m_2,s_2]} e^{-\frac{\pi is}{2}(S_1-S_1'+s_1+S_2-S_2'+s_2)}\times$$
$$\times\,\delta^{(2)}(S_1-S_1'+s_1)\,\delta^{(2)}(S_2-S_2'+s_2)\,e^{\frac{\pi in}{k+2}(m_1+m_2-\hat{M}+\hat{M}')}\times$$
$$\times\sum_l\Big(N^l_{LL'}N^l_{l_1l_2}\delta^{(2k+4)}(M-M'+m_1-m_2)+$$
$$+(-1)^{n+s}N^l_{LL'}N^l_{l_1k-l_2}\delta^{(2k+4)}(M-M'+m_1-m_2+k+2)\Big)\times$$
$$\times\,\chi_{[l_1,m_1,s_1]}(\tilde{q})\,\chi_{[l_2,m_2,s_2]}(\tilde{q})\,. \qquad \text{(D.6)}$$

In the above equation, the sum runs again over equivalence classes $[l_i,m_i,s_i]$. If we relax this condition and sum over all triples with $l_i+m_i+s_i$ even, we obtain instead

$$\langle\!\langle L',M',\hat{M}',S_1',S_2'\|q^{\frac{1}{2}(L_0+\bar{L}_0)-\frac{c}{12}}\|L,M,\hat{M},S_1,S_2\rangle\!\rangle_{(-1)^{(s+1)F}g^n} =$$
$$= \frac{1}{2}\sum_{(l_1,m_1,s_1),(l_2,m_2,s_2)} e^{-\frac{\pi is}{2}(S_1-S_1'+s_1+S_2-S_2'+s_2)}\times$$
$$\times\,\delta^{(2)}(S_1-S_1'+s_1)\,\delta^{(2)}(S_2-S_2'+s_2)\,e^{\frac{\pi in}{k+2}(m_1+m_2-\hat{M}+\hat{M}')}\times$$
$$\times\sum_l N^l_{LL'}N^l_{l_1l_2}\delta^{(2k+4)}(M-M'+m_1-m_2)\,\chi_{[l_1,m_1,s_1]}(\tilde{q})\,\chi_{[l_2,m_2,s_2]}(\tilde{q})\,. \qquad \text{(D.7)}$$

Since $M+\hat{M}$ is even it is immediate (from the $\delta^{(2k+4)}$ constraint) that $m_1+m_2-\hat{M}-\hat{M}'$ is even. Furthermore, the form of the amplitude shows explicitly that the twist leads to an insertion of the group action in the open string sector, where the action on the Chan-Paton labels is specified by $\hat{M}$ and $\hat{M}'$, respectively.

The overlap between a permutation and a tensor product boundary state (in two factors) is again subtle; taking into account the phase of (5.7) one finds

$$\langle\langle L_1, L_2, \hat{M}', S_1', S_2' \| q^{L_0+\bar{L}_0-\frac{c}{24}} \| L, M, \hat{M}, S_1, S_2 \rangle\rangle_{(-1)^{F(s+1)}g^n} =$$
$$= \sum_{(l',m',s')} \chi_{[l',m',s']}(\tilde{q}^{\frac{1}{2}})\, e^{\frac{\pi i n}{k+2}(\hat{M}'-\hat{M}+m'+1)} e^{-\frac{\pi i s}{2}(S_1+S_2-S_1'-S_2'+s'+1)} \times$$
$$\times\, \delta^{(2)}(S_1-S_1'+S_2-S_2'+s'+1)$$

Here the sum runs again over the triples (l',m',s') such that $l'+m'+s'$ is even. One can easily check that $\hat{M}'-\hat{M}+m'+1$ is always even. Furthermore, as before in the untwisted case (that was discussed in section 4), s' is always odd if the two spin structures of the boundary states are the same. In this case, the representations with s' odd should be interpreted as twisted NS sector representations.

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